# Analytical aspects of relaxation for single-slip models in finite crystal plasticity

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### Zusammenfassung

Moderne mathematische Zugänge zu elasto-plastischem Materialverhalten führen über einen zeitdiskreten Energieansatz zu nicht-konvexen Variationsproblemen, die sich den Standardmethoden der Variationsrechnung entziehen. Diese Arbeit beschäftigt sich mit geometrisch nicht-linearer Kristallplastizität und hierbei speziell mit 2D Modellen mit einem aktiven Gleitsystem. Um Informationen über das makroskopische Verhalten solcher Materialien zu erhalten, wird die Relaxierung der einzelnen Inkrementprobleme untersucht, wobei wir uns hier ausschließlich auf einen Zeitschritt beschränken. Unser Augenmerk liegt dabei auf der Frage, ob auf starrer Elastizität basierende Modelle als gute Approximation für physikalisch realistischere Systeme mit elastischer Energie dienen können, zumindest falls letztere elastische Verzerrung energetisch hinreichend stark bestrafen. Die interessante Entdeckung ist nun, dass die Antwort entscheidend davon abhängt, ob man ein Modell mit oder ohne Verfestigung betrachtet.

Unter Berücksichtigung linearer Verfestigung bekommt man ein im obigen Sinne positives Ergebnis, das mathematisch mittels $\Gamma$-Konvergenz untermauert wird. Der Beweis der Kompaktheit und der unteren Schranke stützt sich einerseits auf sorgfältige algebraische Abschätzungen, die die anisotrope Struktur des Problems erfassen, und andererseits auf eine geschickte Verallgemeinerung des klassischen div-curl Lemmas, die den Grenzübergang in der Inkompressibilitätsbedingung ermöglicht. Für die Konstruktion einer den $\Gamma$-Grenzwert realisierenden Folge verwenden wir lokale Laminate mit ortsabhängigen Perioden und Orientierungen.

Sobald im Modell jedoch auf Verfestigung verzichtet wird, verschwindet die zugehörige relaxierte Energiedichte vollständig auf einer großen Menge von Verformungsgradienten, was auf das sublineare Wachstum der Energiedichte zurückzuführen ist. In diesem Fall kann also nicht von einer guten Approximation zwischen dem starren Modell und dem mit elastischer Energie gesprochen werden. Aus physikalischer Sicht bewirkt das Fehlen von Verfestigung auf mikroskopischer Skala die Bildung von Mikrostrukturen und makroskopisch eine extrem weiche Materialantwort auf eine große Klasse externer Kräfte.

## Abstract

Modern mathematical approaches to plasticity lead to non-convex minimization problems for which the standard methods of the calculus of variations are not applicable. In this thesis we consider geometrically nonlinear crystal elastoplasticity in two dimensions with one active slip system. In order to derive information about macroscopic material behavior the relaxation of the single incremental problems, which result from the applied time-discrete variational approach, needs to be investigated. Here our studies are restricted to the first time step only. We especially focus on the question of whether realistic systems with an elastic energy leading to large penalization of small elastic strains can be well-approximated by models based on the assumption of rigid elasticity. The interesting finding is that there are qualitatively different answers depending on whether hardening is included or not.

In presence of linear hardening we obtain a positive result, which is mathematically backed up by $\Gamma$-convergence. The proof of compactness and the lower bound relies on careful algebraic estimates capturing the anisotropic structure of the problem and on a subtle generalization of the classical div-curl lemma to recover incompressibility in the limit. For the construction of a recovery sequence we use local laminates with position-dependent period and orientation.

In the case without hardening, however, the associated relaxed energy density can be shown to vanish for a large class of applied loads, which is due to the sublinear growth of the energy density. Consequently, the desired relation between the rigid model and the one with elastic energy does not hold here. Physically speaking, absence of hardening implies formation of microstructure and very soft macroscopic behavior of the sample in response to a wide range of external forces.

# Contents

# 1. Introduction

Experimental observations reveal a wide range of solid materials developing microstructure under the impact of external forces originated by stress, strain or electromagnetic fields. For the purpose of this thesis any substructure on a scale between the atomic and the macroscopic level is referred to as microstructure. In fact, fine structures can generate fascinating patterns possibly ranging over several length scales and they are often the cause of astonishing material features with regard to strength, rigidity, ductility, hardness or temperature depending behavior. Owing to their particular physical properties many of these materials are used in industrial applications and are therefore of practical relevance. They include micromagnetic and elastoplastic materials, shape memory alloys, which result from solid-solid phase transitions, and nematic elastomers to state only a few examples. Over the last decades there has been a lot of research in the fields of experimentally oriented physics, engineering science and applied mathematics concerned with the questions of how microstructures emerge, what they look like and in what way they influence the performance of the whole body.

Formation of fine-scale structure can basically result from either an inhomogeneous arrangement of material components, as it is for instance the case with grains in polycrystals and mixtures of micromagnetic materials, or from a substantial lack of convexity in the relevant energy density leading to non-existence of classical minimizing states. The latter suggests an underlying continuum model based on an energetic formulation and hence a variational approach. One of the great advantages of this kind of modeling is that all the arising multiscale phenomena can be completely explained by the fundamental principle of energy minimization, which seems very natural. Moreover, the calculus of variations has to offer a large number of highly-developed techniques for the analysis of variational problems, so that a wide base of mathematical tools is available. So far variational models have been successfully applied to the study of material behavior in different fields including the above mentioned examples, see [29, 14, 7, 8, 30] and the references therein.

In general, fine-scale oscillations cannot be resolved exactly, for numerical calculations are far too expensive and therefore out of reach. Besides, mere computation does not provide qualitative insight into the underlying mechanisms. Therefore it is so important to find a way to capture the influence of microstructure on macroscopic material response without knowing every single detail of the behavior on fine scales. To this end the mathematical theory of relaxation was established by Morrey [60] and Dacorogna [25]. It relies on the assumption that large-scale effects can be modeled by optimizing

the prevailing energy locally with respect to all admissible microstructures. Technically speaking, this amounts to determining the corresponding quasiconvex envelope, which in turn means solving an infinite-dimensional, nonlinear optimization problem. However, up to now a rigorous analytical quasiconvexitication of energy densities has only been possible in very few quite specific cases, [24, 45, 28] and a satisfactory mathematical theory for numerical computation is not yet available, even if there is permanent effort in this direction, [13, 15]. So pushing on technological progress concerning this problem is certainly a rewarding challenge.
The class of materials we want to center on in this thesis are crystals. More precisely, we would like to take a further step in understanding the behavior of single-crystal specimens under complex loading conditions such as tension, shear, torsion, etc. Crystalline states, which are very common in nature, just to mention the various types of metals, are characterized by the regular order of atoms in a spatial lattice. The different directions within the lattice structure are not equivalent in response to elastoplastic deformations. But instead discrete symmetries occur, so that one has to expect anisotropic material response. In crystal plasticity plastic flow is due to the formation of glide bands and generated by the movement of dislocation lines along defined slip systems.
Fundamental aspects of modeling finite-strain deformations of elastoplastic material trace back to Lee [48], Rice [68], Kröner [47]. These were later extended and improved by Ortiz and Repetto [67], Carstensen, Hackl and Mielke [14], Miehe, Schotte and Lambrecht [52] and Aurby and Ortiz [6] amongst others. The common key idea of these references is to formulate the rate-independent evolution of elastoplastic bodies within a time-discrete variational setting. To be more specific, in every time increment, which results from an appropriate discretization, a minimization problem has to be studied. Let us point out that the internal variables associated to each of these problems depend essentially on the solution of the previous time step in order to account for deformation history and its accompaniments, such as hardening. Finally, one ends up with a sequence of variational problems resembling those of nonlinear elasticity. For a discussion of the interesting but sophisticated issue of a time-continuous limit for this model we point the reader to recent papers by Mielke, e.g. [53, 54]. In this work, though, we want to stick to the time-discrete version of the evolution problem.
Despite of a profound basis regarding the development of suitable geometrically nonlinear models for the phenomena observed in elastoplasticity, extensive studies, especially within the mathematical framework, are still in the early stages. In keeping with this line of thought the present thesis is devoted to the investigation of macroscopic material behavior for a single-crystal model with one active slip system involving large penalization of elastic strains. Our intention is to derive useful information about the relaxation of this system by putting it in relation to the corresponding elastically rigid model, which is one of the few examples with explicitly known effective energy [24, 18]. Note that the most important results presented in the following have already been published [20] or are publications in preparation [21].

**A single-slip model with elastic energy.** To go a bit more into detail, this thesis is based on the time-discrete variational approach as it was introduced by Ortiz and Repetto in [67] and by Carstensen, Hackl and Mielke in [14]. In what follows we assume an experimental setting with a monotone loading path and we limit ourselves to the single minimization problem of the first incremental step.

Let $\Omega$ be the reference configuration of a two-dimensional elastoplastic body and $u : \Omega \to \mathbb{R}^2$ its total deformation (in the first time step). The term $F = \nabla u$ is called the deformation gradient. Accounting for finite strains naturally yields a multiplicative decomposition of $F$ into an elastic part $F_{\mathrm{el}}$ and a plastic one $F_{\mathrm{pl}}$, i.e. $F = F_{\mathrm{el}}\, F_{\mathrm{pl}}$. We also use the common assumption that plastic deformations are volume preserving, meaning $\det F_{\mathrm{pl}} = 1$. The system energy in a single time increment, which has to be minimized, is supposed to consist of three components, that is

$$\int_\Omega W_{\mathrm{el}}(F_{\mathrm{el}}) + W_{\mathrm{pl}}(F_{\mathrm{pl}}) + \mathrm{Diss}(F_{\mathrm{pl}})\, \mathrm{d}x\,,$$

where $W_{\mathrm{el}}$ corresponds to the elastic effects and $W_{\mathrm{pl}} + \mathrm{Diss}$ stands for the energetic contributions of plastic deformation such as hardening or energy dissipation. Throughout this work we assume only one slip system is active and that it is characterized by two orthogonal vectors, the slip direction $s \in \mathbb{S}^1$ and the slip-plane normal $m \in \mathbb{S}^1$. Further, $\gamma$ denotes the slip strain along $(s, m)$. Then, the plastic energy density reads

$$W_{\mathrm{pl};p}(F_{\mathrm{pl}}) + \mathrm{Diss}_p(F_{\mathrm{pl}}) = \begin{cases} |\gamma|^p & \text{for } \; F_{\mathrm{pl}} = \mathbb{I} + \gamma s \otimes m\,, \\ \infty & \text{else} \end{cases}$$

with $p \in \{1, 2\}$. Here the plastic exponent $p = 1$ stands for the case without hardening, while linear hardening is included into the model by setting $p = 2$. For simplicity we choose the elastic energy density with quadratic growth

$$W_{\mathrm{el},\varepsilon}\,(F_{\mathrm{el}}) = W_{\mathrm{el},\varepsilon;2}\,(F_{\mathrm{el}}) = \frac{1}{\varepsilon}\,\mathrm{dist}^2\big(F_{\mathrm{el}}, \mathrm{SO}(2)\big), \tag{1.1}$$

where the parameter $\varepsilon > 0$ is taken to be small. In view of physical interpretation $W_{\mathrm{el},\varepsilon}$ penalizes elastic deformations differing from rigid body motions and $\varepsilon$ symbolizes the ratio of critical stress and elastic constants. After adding the above energy densities and optimizing over all possible decompositions of the deformation gradient $F$ one eventually obtains the condensed energy density

$$W_\varepsilon(F) = W_{\varepsilon;2,p}(F) = \inf_{\gamma \in \mathbb{R}} \big\{ \frac{1}{\varepsilon}\,\mathrm{dist}^2\big(F\,(\mathbb{I} - \gamma\; s \otimes m)\,, \mathrm{SO}(2)\big) + |\gamma|^p \big\},$$

which is going to be the object of our main interest. It is important to observe that the procedure of combining the multiplicative decomposition of $F$ with the pointwise minimization over the internal variables $F_{\mathrm{pl}}$ and $\gamma$ results in non-standard growth for $W_\varepsilon$.

**An elastically rigid model with one active slip system.** Considering the limit $\varepsilon \to 0$ in (1.1) provides a new elastic energy density for which only rigid body rotations are admissible. As a consequence one obtains a simplified single-slip model capturing elastically rigid material behavior. Conti and Theil [24, 18] were the first to analyze the associated condensed energy potential

$$W(F) = W_p(F) = \begin{cases} |\gamma|^p & \text{for } F \in \mathcal{M}^{(2)}, \\ \infty & \text{else,} \end{cases}$$

with $p \in \{1,2\}$, where $\mathcal{M}^{(2)} = \{F \in \mathbb{R}^{2\times 2}|\ F = R(\mathbb{I} + \gamma s \otimes m),\ R \in \mathrm{SO}(2),\ \gamma \in \mathbb{R}\} = \{F \in \mathbb{R}^{2\times 2}|\ \det F = 1,\ |Fs| = 1\}$. They even managed to explicitly determine its relaxation by proving a representation formula for the quasiconvex envelope of $W$, namely

$$W_1^{\mathrm{qc}}(F) = \begin{cases} \sqrt{|F|^2 - 2} & \text{for } F \in \mathcal{N}^{(2)}, \\ \infty & \text{else,} \end{cases} \tag{1.2}$$

and

$$W_2^{\mathrm{qc}}(F) = \begin{cases} |Fm|^2 - 1 & \text{for } F \in \mathcal{N}^{(2)}, \\ \infty & \text{else,} \end{cases} \tag{1.3}$$

where the set $\mathcal{N}^{(2)} = \{F \in \mathbb{R}^{2\times 2}|\ \det F = 1,\ |Fs| \le 1\}$ is the quasiconvex hull of $\mathcal{M}^{(2)}$. In short, under the hypothesis of rigid elasticity macroscopic effects are entirely characterized by (1.2), (1.3) and therefore well-understood.

**Aims and central questions.** Our goal for this work is to apply or if necessary develop the right mathematical tools and concepts which will enable us to predict material behavior for the models under consideration. Here the crucial questions are the following: Firstly, can the knowledge about the features of the rigid model help in some way to derive novel information about the models with elastic energy? This immediately raises a second question. Is there a close relation between the two types of models at all, meaning between the one resulting from sufficiently large penalization of elastic deformations and the one based on rigid elasticity? In particular, this includes the query for a suitable approximation between the effective energy densities for small $\varepsilon$ to guarantee similar material response on a macroscopic scale.
The next two paragraphs give a brief overview of what answers to expect. It is remarkable that the findings depend highly on whether the considered models involve hardening or not.

**The model without hardening.** If no hardening is involved, meaning $p = 1$, the crucial observation is that there are curves along which $W_{\varepsilon;2,1}$ grows merely sublinearly. This is probably not what one would expect at first glance. By considering for example

$t \to F_t = (t\mathbb{I})(\mathbb{I} + t^2 s \otimes m)$ we find

$$W_{\varepsilon;2,1}(F_t) \leq \frac{c}{\varepsilon}|F_t|^{2/3}$$

with a constant $c > 0$. Exploiting such directions for a subtle construction of an appropriate family of rank-one lines gives rise to this astonishing theorem taken from [20].

**Theorem 1.1** *For $\varepsilon > 0$ it holds*

$$W^{\mathrm{qc}}_{\varepsilon;2,1}(F) = 0 \qquad \textit{for all } F \in \mathcal{N}^{(2)}.$$

In words, the relaxed energy density of $W_{\varepsilon;2,1}$ vanishes identically on $\mathcal{N}^{(2)}$ for all $\varepsilon > 0$, while $W_1^{\mathrm{qc}}$ does not, compare (1.2). Consequently, $W_1$ cannot be seen as a good approximation to $W_{\varepsilon;2,1}$, even if the pointwise limit $\lim_{\varepsilon\to 0} W_{\varepsilon;2,1}(F) = W_1(F)$ for all $F \in \mathbb{R}^{2\times 2}$ might suggest the opposite. In terms of physics these findings reveal microstructure formation and very soft material behavior with vanishing macroscopic stress in response to a large class of applied loads.

**The model with linear hardening.** As soon as we assume presence of linear hardening ($p = 2$), however, the situation is qualitatively different. In contrast to the previous case the regularizing effect of hardening renders an approximation result via $\Gamma$-convergence possible. In fact, the elastically rigid model turns out to determine the $\Gamma$-limit of the models with an increasing penalization of elastic strains. In order to give a mathematically precise formulation of the indicated finding, we state the following theorem.

**Theorem 1.2** *Let $X = \{u \in W^{1,1}(\Omega;\mathbb{R}^2) \mid \int_\Omega u = 0\}$ be endowed with the strong $L^1$-topology. For $\varepsilon > 0$ we define the energy functionals $E_{\varepsilon;2,2},\, E_2 : X \to \overline{\mathbb{R}}$ by*

$$\begin{aligned} E_{\varepsilon;2,2}[u] &= \int_\Omega W_{\varepsilon;2,2}(\nabla u)\,\mathrm{d}x, \\ E_2[u] &= \begin{cases} \int_\Omega W_2^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x, & \textit{if } u \in W^{1,2}(\Omega;\mathbb{R}^2)\cap X,\ \nabla u \in \mathcal{N}^{(2)} \textit{ a.e. in } \Omega, \\ \infty, & \textit{otherwise.} \end{cases} \end{aligned}$$

*Then, $E_{\varepsilon;2,2}$ converges to $E_2$ in $X$ as $\varepsilon \to 0$ in the sense of $\Gamma$-convergence and the corresponding compactness result holds.*

The proofs of compactness and the lower bound are widely based on algebraic estimates exploiting the special anisotropic structure of $W_{\varepsilon;2,2}$. The crucial point, though, is the recovery of the incompressibility constraint in the limit. For this purpose Conti, Dolzmann and Müller [22] recently developed a subtle compensated compactness result. It generalizes the classical div-curl lemma in the sense that div and curl of the two $L^2$-weakly converging sequences are only required to be compact as functionals on Lipschitz functions provided their product is equi-integrable. Concerning the construction of the recovery sequence we use local laminates whose rank-one gradients and periods depend essentially on the position in $\Omega$.

**Outline of the text.** This thesis is arranged as follows. Right after these introductory words we start by giving some physical background and motivation in terms of application. For that purpose we provide a discussion of finite plasticity and the time-discrete variational formulation of elastoplasticity. Chapter 2 is concluded by a section on simple single-crystalline models with special focus on single-slip systems.
In Chapter 3 the fundamental mathematical theories underlying the proof of the above-mentioned results are introduced. One major issue is the concept of relaxation. The goal of the latter is to determine effective energy densities governing macroscopic material behavior, which is closely related to computing quasiconvex envelopes. In this context we recall the definition of the different notions of convexity and comment on their main properties and relations. Another important method is $\Gamma$-convergence, which is addressed here, too.
As already said, special single-slip models with elastic energy are at the heart of the present thesis. In Chapter 4 we establish these models in detail using general growth exponents for both the elastic and the plastic energy contribution. Besides, we state the features of the corresponding elastically rigid models whose macroscopic behavior is by now completely understood [24, 18].
We continue in Chapter 5 by summarizing some technical tools and useful lemmata from different fields with topics ranging from results on separately convex functions and trace and embedding theorems for Sobolev and Besov functions, over the famous Caldéron-Zygmund theorem on singular integrals, to corollaries of the convex integration method by Müller and Šverák, [63, 64].
The main focus of Chapter 6, where we collect important auxiliary results, is on an alternative proof of a generalized version of the classical div-curl lemma by Conti, Dolzmann and Müller [22]. Here we employ Helmholtz-decomposition and elliptic $L^p$-theory instead of basing the argumentation on Lipschitz truncations, as the authors proceed in [22]. With this compensated compactness result at hand one has gathered all the means for setting out to prove the main findings of this thesis.
That is eventually done in Chapter 7. But before that we state the postulated results, Theorem 1.1 and Theorem 1.2, again at full length and comment on their classification and special features. Besides, for the case with linear hardening and quadratic elastic energy we show some specific properties of the effective energy densities of $W_{\varepsilon;2,2}$ including regularity and pointwise convergence to $W_2^{\mathrm{qc}}$ as $\varepsilon$ tends to zero. Next the reader will find different generalizations, such as the investigation of more general plastic and elastic growth exponents, the application of the developed theory to realistic elastic energy densities and a brief discussion on incorporating boundary conditions. We complete our studies of the two-dimensional situation by giving a more elementary proof of compactness in the $\Gamma$-convergence result provided there is linear hardening and the elastic energy

density displays growth of order four.
Chapter 8 is concerned with the three-dimensional case and deals with the following questions. To what extent can the findings of the 2D setting be modified to obtain similar results here and where are the fundamental differences?
Finally we close with a brief outlook in Chapter 9, where a number of possible future projects and directions for further research are pointed out.
Additionally, Appendix A contains information about the mathematical notation in this thesis and gives a tabular overview of the expressions and symbols used throughout the text.


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# 2. An outline of finite crystal plasticity

The first step in understanding a physical phenomenon is usually to develop an effective model for the situation at hand. In terms of materials science plasticity is, roughly speaking, the material property of performing irreversible changes in shape responding to external forces. In fact, it turned out that the theory of finite plasticity, which came up in the 1960s of the last century [47, 48, 49], is a suitable concept for modeling the behavior of elastoplastic material under all kinds of deformations, such as bending, squeezing, pulling, shearing, twisting, etc. Let us remark that finite strain plasticity is often called geometrically nonlinear plasticity and stands in contrast to the classical theory, which is restricted to infinitesimal strains only and is essentially a linear theory.

## 2.1. Basic facts on finite plasticity

The reference configuration of an elastoplastic body is modeled by a set $\Omega \subset \mathbb{R}^n$ with space dimension $n = 2, 3$. Then the function $u : [0, T] \times \Omega \to \mathbb{R}^n$ with $T > 0$ describes the time-dependent total deformation of the sample and $F = \nabla u$ is called the deformation gradient. Taking finite strains into account naturally leads to a multiplicative decomposition of $F$ into an elastic part $F_{\text{el}}$ and a plastic one $F_{\text{pl}}$, i.e.

$$F = F_{\text{el}} \, F_{\text{pl}}. \tag{2.1}$$

More precisely, $F_{\text{pl}}$ codes the irreversible deformation and the history of plastic flow, while $F_{\text{el}}$ stands for the current elastic stress.

Since equality (2.1) is the fundamental principle in geometrically nonlinear plasticity, we think it is worth saying a word on the main idea and the physical motivation behind this postulate. Here we follow the lines of [48] and provide Figure 2.1 for illustration.

Imagine an elastoplastic body subdivided into single particles at any state of deformation. Let $x$ denote an element of the sample in the reference configuration and $u(t, x)$ the corresponding particle of the deformed state at some time $t \in [0, T]$. Now, if the present surface forces are suspended, so that every element is unstressed, one obtains an intermediate configuration with the element $u_{\text{pl}}(t, x)$ associated to $x$. Consequently, this stress-free configuration results from a purely plastic deformation, since the elastic strain components have been released. Locally, meaning in the neighborhood of a particle, the deformation $u$ can be suitably expressed in terms of its linear approximate, the deformation gradient $F$. The same holds true for $u_{\text{pl}}$ and $F_{\text{pl}} = \nabla u_{\text{pl}}$, as well as for the

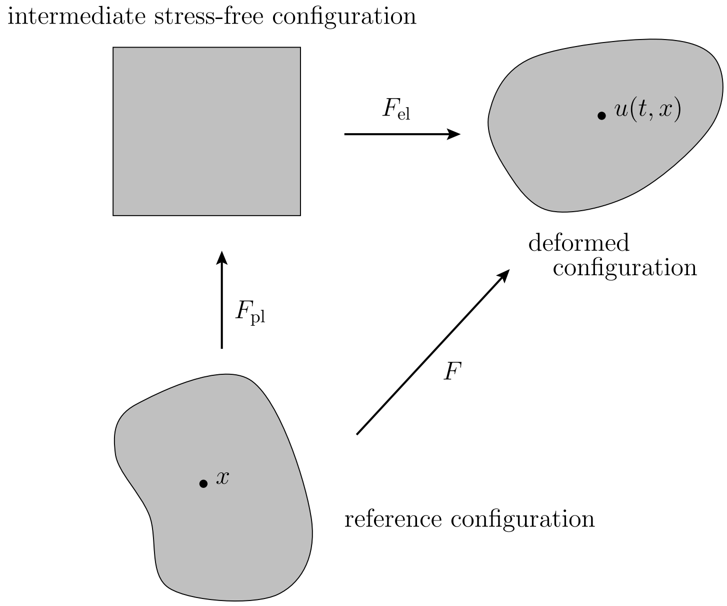


Figure 2.1.: Multiplicative decomposition of the deformation gradient

mapping from the intermeditate to the deformed configuration $u_{\mathrm{el}}$, which takes $u_{\mathrm{pl}}(t,x)$ to $u(t,x)$, and $F_{\mathrm{el}} = \nabla u_{\mathrm{el}}$ provided these two functions are sufficiently smooth. Then the multiplicative, non-commutative split (2.1) is deduced from $u = u_{\mathrm{el}} \circ u_{\mathrm{pl}}$ using the chain rule of differentiation. In fact, however, the maps $u_{\mathrm{pl}}$ and $u_{\mathrm{el}}$ are in general not even continuous, because they are only unique up to rigid body rotations, which can be different for every single particle. In this case it is not possible to define $F_{\mathrm{el}}$ and $F_{\mathrm{pl}}$ as above. Nevertheless one can regard the two expressions in terms of local linear mappings that occur in the limit of small material particles and still fulfill (2.1), but neither the formulas $F_{\mathrm{el}} = \nabla u_{\mathrm{el}}$, $F_{\mathrm{pl}} = \nabla u_{\mathrm{pl}}$ nor the associated continuity requirement. In particular, $F_{\mathrm{el}}$ and $F_{\mathrm{pl}}$ need not be gradients any more.
Note that the deformation $u$ itself does not provide a unique characterization of the decomposition (2.1). Practically speaking, you can never know the whole deformation history of a body from simply looking at it at a certain state of deformation. Therefore one has to introduce a suitable set of internal variables which uniquely determines the multiplicative relation between $F$, $F_{\mathrm{el}}$ and $F_{\mathrm{pl}}$. For this purpose one usually takes the plastic deformation gradient $F_{\mathrm{pl}}$ together with a vector $p \in \mathbb{R}^M$, which is closely linked to the mechanical properties of the material by characterizing for instance hardening behavior. In this case $p$ is referred to as hardening variables or parameters. Besides, we use the common assumption that plastic deformation does not imply volume changes

and thus require $\det F_{\mathrm{pl}} = 1$. Then, $(F_{\mathrm{pl}}, p) \in \mathrm{Sl}(n) \times \mathbb{R}^M$ is the set of internal variables of the system. In the following we will sometimes switch to the notation $P = F_{\mathrm{pl}}^{-1}$, which is well-defined by the postulated incompressibility, and take $P$ as an internal variable in place of $F_{\mathrm{pl}}$, so that the set of plastic variables reads $z = (P, p)$. This procedure is quite frequent in the engineering literature.
Before we continue with the issue of evolution of elastoplastic bodies, let us briefly explain the essential difference between the geometrically nonlinear and the classical theory of plasticity. Unlike the product relation of (2.1) linearized elastoplasticity is based on the additive formula $\varepsilon = \varepsilon_{\mathrm{el}} + \varepsilon_{\mathrm{pl}}$, where the strain tensor $\varepsilon$ is the symmetric part of the gradient of the displacement $v : [0, T] \times \Omega \to \mathbb{R}^n$, meaning $\varepsilon = \frac{1}{2}(\nabla v + (\nabla v)^T)$. For these linear concepts there is by now a complete mathematical theory available, which was established in the 1970s, and was later enhanced and refined up to the development of numerical algorithms, see for instance [43]. This book and the references therein give an extensive review of the subject both from an analytical and numerical viewpoint.

## 2.2. Rate-independent evolution of elastoplastic bodies

The investigation of how elastoplastic bodies are effected by external time-dependent loading is mathematically speaking an evolution problem. Let us now present a time-discrete variational approach which helps to obtain approximate solutions to the actually time-continuous problem. Here we apply the incremental method for rate-independent processes leading to a sequence of minimization problems, as it can be found in [67, 52, 54, 14, 42, 53]. In the literature there are basically two different ways of dealing with dissipation in the energetic formulation. These will be introduced and briefly discussed in this paragraph.
Notice that rates or in other words partial derivatives with respect to time are always denoted by $\dot{( )}$.

### Variational formulation based on the flow rule

The first concept relies on the plastic flow rule, which is derived from the fundamental principle of maximum plastic dissipation. In this section we mainly proceed along the lines of [14, 67]. As before, let $P$ be the inverse of the plastic deformation gradient and $p \in \mathbb{R}^M$ the hardening parameters. The stored energy density $\psi$ is supposed to depend on $F_{\mathrm{el}} = FP$ and $p$ only, meaning

$$\psi(F, P, p) = \overline{\psi}(FP, p) = \overline{\psi}(F_{\mathrm{el}}, p).$$

The motivation for this assumption is that pure plastic deformations are observed to result from a recast of the original material configuration into another and hence do not affect the elastic properties. The internal plastic variable $p$, though, detects changes in $P$

by recording the history of deformation and may therefore influence the elastic behavior. Further, the energy density $\psi$ is taken to be frame-indifferent, i.e. $\overline{\psi}(RF_{\mathrm{el}}, p) = \overline{\psi}(F_{\mathrm{el}}, p)$ for all $R \in \mathrm{SO}(n)$, and coercive in the sense that $\overline{\psi} \to \infty$, if $|F_{\mathrm{el}}| + |F_{\mathrm{el}}^{-1}| + |p| \to \infty$. Note that we restrict our outline to homogenous materials, but it is immediate to see that the theory still applies, if all the relevant quantities additionally depend on $x \in \Omega$, e.g. $\psi(x, F, P, p) = \overline{\psi}(x, F_{\mathrm{el}}, p)$.
The thermo-mechanical dual variables corresponding to $F$, $P$ and $p$ are the first Piola-Kirchhoff stress tensor

$$T = \frac{\partial}{\partial F}\psi(F, P, p) = \frac{\partial}{\partial F_{\mathrm{el}}}\overline{\psi}(F_{\mathrm{el}}, p)P^T,$$

the conjugate plastic stresses

$$Q = -\frac{\partial}{\partial P}\psi(F, P, p) = -F^T\frac{\partial}{\partial F_{\mathrm{el}}}\overline{\psi}(F_{\mathrm{el}}, p)$$

and the conjugate hardening forces

$$q = -\frac{\partial}{\partial p}\psi(F, P, p) = -\frac{\partial}{\partial p}\overline{\psi}(F_{\mathrm{el}}, p).$$

Besides, we observe that the quantity $\overline{Q} = P^T Q$ depends on $F_{\mathrm{el}}$ and $p$, while being independent of $P$.
In order to describe the evolution of $(P, p)$ an appropriate quantity for the characterization of the threshold between plastic and elastic material behavior is needed. To this end we choose the yield function $\varphi = \varphi(T, Q, q)$ and postulate that $\varphi$ takes the form $\varphi(T, Q, q) = \overline{\varphi}(\overline{Q}, q)$. Apart from that it is required to fulfill the necessary properties, so that the set of admissible stresses

$$\mathbb{Q} = \left\{(\overline{Q}, q) \in \mathbb{R}^{n\times n} \times \mathbb{R}^M \mid \overline{\varphi}(\overline{Q}, q) \le 0\right\}$$

is closed and convex with $(0, 0) \in \mathbb{Q}$. By $\chi$ let us denote the characteristic function of $\mathbb{Q}$ defined as $\chi(\overline{Q}, q) = 0$, if $(\overline{Q}, q) \in \mathbb{Q}$ and $\chi(\overline{Q}, q) = \infty$ otherwise.
The principle of maximal dissipation, which is a consequence of the second law of thermodynamics, says that the dissipation caused by plastic deformation, in formulas

$$-\frac{\partial}{\partial P}\psi(F, P, p) : \dot{P} - \frac{\partial}{\partial p}\psi(F, P, p) : \dot{p} = Q : \dot{P} + q \cdot \dot{p},$$

has the maximal value

$$\sup_{(P^T Q, q)\in\mathbb{Q}} \{Q : \dot{P} + q \cdot \dot{p}\} = \sup_{(\overline{Q}, q)\in\mathbb{Q}} \{\overline{Q} : (P^{-1}\dot{P}) + q \cdot \dot{p}\}$$

for fixed $P^{-1}\dot{P}$ and $\dot{p}$. According to a derivation tracing back to [58], this basic principle gives rise to the flow rule in the form

$$\left(P^{-1}\dot{P},\dot{p}\right) = \lambda\left(\frac{\partial\varphi}{\partial\overline{Q}}(\overline{Q},q),\frac{\partial\varphi}{\partial q}(\overline{Q},q)\right) \tag{2.2}$$

for $\varphi$ and $\lambda$ satisfying the complementarity condition $\varphi \leq 0 \leq \lambda$ and $\lambda\varphi = 0$. In view of the definition of $\chi$ and its Legendre transform

$$\chi^*(S,s) = \sup_{(\overline{Q},q)\in\mathbb{R}^{n\times n}\times\mathbb{R}^M}\{\overline{Q}:S+q\cdot s-\chi(\overline{Q},q)\} = \sup_{(\overline{Q},q)\in\mathbb{Q}}\{\overline{Q}:S+q\cdot s\}$$

with $(S,s)\in\mathbb{R}^{n\times n}\times\mathbb{R}^M$, both the flow rule and the expression of maximal dissipation can be reformulated. The latter is what we actually call the dissipation of the system and it reads $\chi^*(P^{-1}\dot{P},\dot{p})$ with $\chi^*$ being obviously positively 1-homogenous and non-negative, since $0\in\mathbb{Q}$. Concerning the flow rule one immediately obtains the alternative formulations

$$(P^{-1}\dot{P},\dot{p})\in\partial\chi(\overline{Q},q) \quad \text{or equivalently} \quad (\overline{Q},q)\in\partial\chi^*(P^{-1}\dot{P},\dot{p}), \tag{2.3}$$

where $\partial\chi$ and $\partial\chi^*$ are the subdifferentials of the convex functions $\chi$ and $\chi^*$, respectively. The equivalence in (2.3) follows directly from convex analysis, in particular from the well-known relation between Legendre transform and subdifferential.
Next we consider the time-discretization $0=t^0<t^1<\ldots<t^N=T$. Let $(u^0,P^0,p^0)$ be a stable initial state and $(u^k,P^k,p^k)$ with $k\in\{1,\ldots,N\}$ the state variables at time $t^k$. For $k\in\{0,\ldots,N\}$ we prescribe the boundary conditions $u^k=u_b(t^k)$ on $\partial\Omega$ with given $u_b:[0,T]\times\Omega\to\mathbb{R}^n$. Further, time-dependent external loading is modeled by $l$ through $\langle l(t),u\rangle=\int_\Omega f(t)u\,\mathrm{d}x+\int_{\partial\Omega}g(t)u\,\mathrm{d}S$, where $f$ and $g$ stand for the applied body and surface forces, respectively. Before we can pass to the effective variational formulation in each time step the time derivatives $\dot{p}$ and $P^{-1}\dot{P}$ appearing in the dissipation term have to be discretized first. So instead of $\dot{p}$ we take the classical quotient $\frac{p^k-p^{k-1}}{t^k-t^{k-1}}$ in the $k$-th time step. With respect to $P^{-1}\dot{P}$ there are several different possibilities of approximation, so the reader is referred to [14, Chapter 3] for more details. Here we choose $\frac{1}{t^k-t^{k-1}}\big(\mathbb{I}-(P^k)^{-1}P^{k-1}\big)$.
With these preliminaries at hand the functional to be minimized in time step $k$ can be derived as

$$E^k[u,P,p]=\int_\Omega\overline{\psi}\big((\nabla u)P,p\big)+\chi^*\big(\mathbb{I}-(P)^{-1}P^{k-1},p-p^{k-1}\big)\,\mathrm{d}x-\langle l(t^k),u\rangle, \tag{2.4}$$

see [14, Chapter 4] for the exact calculation. The incremental problem is then formulated as follows:

$$\begin{aligned}&\text{For } k=1,\ldots,N \text{ find } u^k:\Omega\to\mathbb{R}^n \text{ with } u^k=u_b(t^k) \text{ on } \partial\Omega \text{ and}\\&(P^k,p^k):\Omega\to\mathrm{Sl}(n)\times\mathbb{R}^M \text{ which minimize } E^k[u,P,p].\end{aligned} \tag{2.5}$$

Since this approach essentially underlies the physical principle of energy minimization and since the integrand in $E^k$ is independent of derivatives of the internal variables by the choice of a fixed discretization, we can minimize the internal variables out pointwise to get the reduced or condensed energy density

$$\psi_{\text{cond}}(\widetilde{P}, \tilde{p}; F) = \inf_{(P,p)\in \mathrm{Sl}(n)\times\mathbb{R}^M} \left\{ \overline{\psi}\,(FP, p) + \chi^*\big(\mathbb{I} - P^{-1}\widetilde{P}, p - \tilde{p}\big) \right\}$$

for given $(\widetilde{P}, \tilde{p}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^M$ and all $F \in \mathbb{R}^{n\times n}$. This defines the corresponding reduced functionals

$$E^k_{\text{cond}}[u] = \int_\Omega \psi_{\text{cond}}\big(P^{k-1}, p^{k-1}; \nabla u\big)\ \mathrm{d}x - \langle l(t^k), u\rangle,$$

for $k \in \{1, \dots, N\}$. Notice that the functionals $E^k_{\text{cond}}$ resemble the variational functionals typically occurring in nonlinear elasticity.
Summing up, in order to obtain an approximate solution for the evolution of an elastoplastic body a sequence of minimization problems of nonlinear elastic type has to be solved.

#### Variational formulation based on dissipation distances

The second theory, for which we mainly follow [53, 54], is a more mathematically oriented method and is built upon an energetic formulation with the elastic energy storage potential $\psi$ and the dissipation potential $\Delta$ as constitutive functions.
Here again we suppose $\psi(F, P, p) = \overline{\psi}(FP, p)$ with the properties pointed out in the previous paragraph. If $l$ stands for the applied external loading, we obtain the stored energy functional

$$\mathcal{E}\big(t, u, (P, p)\big) = \int_\Omega \overline{\psi}\big((\nabla u)P, p\big)\ \mathrm{d}x - \langle l(t), u\rangle.$$

As a second ingredient, alternatively to the flow rule, one can introduce a suitable measure for the energy dissipated when passing from one internal state $\hat{z} = (\hat{P}, \hat{p})$ to another, $\tilde{z} = (\widetilde{P}, \tilde{p})$. To this end we come up with a dissipation potential $\Delta(z, \dot{z}) = \Delta(P, p, \dot{P}, \dot{p}) = \overline{\Delta}(p, P^{-1}\dot{P}, \dot{p})$ encoding internal dissipational friction forces, which in turn can be reobtained by differentiating $\Delta$ with respect to the plastic rates. Then, $\Delta$ is assumed to be convex and positively 1-homogenous with respect $\dot{z}$. This last property implies rate-independence of the model. For a general and detailed introduction to rate-independent processes, see [55]. With the help of $\Delta$ one can define the dissipation distance $D$, which plays the role of a metric on $\mathrm{Sl}(n) \times \mathbb{R}^M$, the space of internal plastic variables. It is given by

$$D(\hat{z}, \tilde{z}) = \inf\big\{ \textstyle\int_0^1 \Delta(z(s), \dot{z}(s))\ \mathrm{d}s \mid z \in C^1([0,1], \mathrm{Sl}(n)\times\mathbb{R}^M),\ z(0) = \hat{z},\ z(1) = \tilde{z} \big\}$$

and satisfies $D(\hat{z},\tilde{z}) = D(\mathbb{I},\hat{p},(\hat{P})^{-1}\widetilde{P},\tilde{p}) = \overline{D}(\hat{p},(\hat{P})^{-1}\widetilde{P},\tilde{p})$. In words, by minimizing the dissipation potential over all sufficiently regular paths between two states of plastic variables one eventually ends up with the desired dissipation distance. Finally, integration in space yields the total dissipation

$$\mathcal{D}(\hat{z},\tilde{z}) = \int_\Omega D(\hat{z}(x),\tilde{z}(x))\,\mathrm{d}x.$$

With the energies $\mathcal{E}$ and $\mathcal{D}$ at hand we can now pass on to the time-discrete variational formulation of the problem. In [53] and the references therein the reader finds a discussion of the time-continuous problem and its relation to the discrete approach in the limit of small time steps, which we will not address in this work.
If we choose the time-discretization $0 = t^0 < t^1 < \ldots < t^N = T$, the initial state $(u^0,z^0) = (u^0,P^0,p^0)$ and boundary data $u_b : [0,T]\times\Omega \to \mathbb{R}^n$ the incremental problems read:

For $k = 1,\ldots,N$ find $u^k : \Omega \to \mathbb{R}^n$ with $u^k = u_b(t^k)$ on $\partial\Omega$ and $z^k = (P^k,p^k) : \Omega \to \mathrm{Sl}(n)\times\mathbb{R}^M$ which minimize

$$E^k[u,z] = \mathcal{E}(t^k,u,z) + \mathcal{D}(z^{k-1},z). \tag{2.6}$$

A step in making the model easier to handle is reduction by pointwise minimization, meaning that we eliminate the internal variables in the elastoplastic energy density by minimizing them out locally. This is possible, because $\overline{\psi}$ and $D$ do not depend on the derivatives of $z$, which is one of the great advantages of this approach. Hence, we gain the condensed energy density

$$\psi_{\text{cond}}(\tilde{z};F) = \min_{z=(P,p)\in\mathrm{Sl}(n)\times\mathbb{R}^M} \left\{\overline{\psi}(FP,p) + D(\tilde{z},z)\right\} \tag{2.7}$$

for $\tilde{z} \in \mathrm{Sl}(n)\times\mathbb{R}^M$ and $F \in \mathbb{R}^{n\times n}$. So the sequence of reduced problems providing approximate solutions for the evolution process under consideration is given as follows:

For $k = 1,\ldots,N$ find $u^k : \Omega \to \mathbb{R}^n$ with $u^k = u_b(t^k)$ on $\partial\Omega$ which minimize

$$E^k_{\text{cond}}[u] = \int_\Omega \psi_{\text{cond}}(z^{k-1};\nabla u)\,\mathrm{d}x - \langle l(t^k),u\rangle, \tag{2.8}$$

where $z^{k-1} \in \operatorname{argmin}\{\overline{\psi}(\nabla u^{k-1}P,p) + D(z^{k-2},z)\}$ for $k > 1$ and $(u^0,z^0)$ is given by the initial data.

In fact, in this thesis we will be content with considering low energy states of (2.8). For the question of existence of minimizers of (2.6) and (2.8) and for a discussion of the necessary conditions on $\psi$, $\psi_{\text{cond}}$ and $\Delta$ to guarantee existence of minimizing states we point e.g.

to [53]. In particular, there is, roughly speaking, the subsequent relation between the two problems, for the correct function spaces and technical details see [54, Chapter 4]. If $(u, z)$ minimizes $E^k$ for some $k \in \mathbb{N}$, then $u$ minimizes $E^k_{\text{cond}}$ and $z \in \operatorname{argmin}\{\overline{\psi}(\nabla u^{k-1}P, p) + D(z^{k-2}, z)\}$. Conversely, for every minimizer $u$ of $E^k_{\text{cond}}$ there exists $z = (P, p)$ such that $(u, z)$ minimizes $E^k$. An investigation of existence of minimizers in incremental finite-strain elastoplasticity incorporating regularizing terms in form of higher gradients can be found in [56].

#### Comparison of the two concepts

While the first theory relying on the flow rule corresponds to what is done in the engineering literature and is closer to applications, the second approach turns out convenient from the mathematical viewpoint, for it provides a profound basis for easier theoretical arguments and simulations of rate-independent processes including the evolution of elastoplastic bodies. The connection between these two ideas was thoroughly analyzed by Mielke, see for instance [53, Section 3.4]. Indeed, there is a one-to-one relation between dissipation potentials and flow rules in form of an explicit formula, which follows from Legendre transformation.

## 2.3. Single-crystal plasticity

The goal of this section is to apply the concepts introduced previously to the context of crystal plasticity. Recall that crystals consist of material whose atoms or molecules are arranged in a periodically repeating pattern, the crystal lattice. The crystals we want to focus on here are predominantly metals. Besides, restricting ourselves to single-crystals helps to avoid complicated grain-boundary effects. In crystalline plasticity, plastic deformation occurs in the form of slip, which takes place along defined slip systems and is caused by the movement of dislocations, which are line defects within the crystal lattice, see e.g. [16] for a detailed discussion of geometrically necessary dislocations.

### 2.3.1. Single-slip models without hardening

Let $\{(s^j, m^j)\,|\, j \in \{1, \ldots, N_s\}\} \subset \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ with $N_s \in \mathbb{N}$ and $s^j \cdot m^j = 0$ be the slip systems of a single-crystal. Here $s^j$ denote the slip directions, while $m^j$ stands for the unit normal to the $j$-th slip plane. A further quantity of importance when modeling plastic behavior is $\tau^j$, the critical resolved shear stress of slip system $j$. In the present setting the vector of internal plastic variables specifies to $\sigma \in \mathbb{R}^{N_s}$ whose components $\sigma_j$ characterize the amount of slip along the respective slip system and the hardening parameters $p \in \mathbb{R}^{M-N_s}$. The time derivative of $\sigma_j$, denoted $\dot{\sigma}_j$, is named slip rate and is supposed to be non-negative. By this postulate we formally distinguish between the two orientations of a slip system. Postponing hardening effects to Section 2.3.2 we may assume for the moment that $(P, \sigma) \in \mathrm{Sl}(n) \times \mathbb{R}^{N_s}$ is the set of internal variables of the system. As before the elastic potential is assumed to depend on the elastic part of the deformation gradient, so that one has $\overline{\psi}(FP, \sigma)$. The classical single-crystal flow rule [68] characterizes the relation between the internal variables $P$ and $\sigma$, namely

$$P^{-1}\dot{P} = \sum_{j=1}^{N_s} \dot{\sigma}_j s^j \otimes m^j. \tag{2.9}$$

The corresponding expression for the dissipation potential was determined in [42] and takes the form

$$\Delta(P, \sigma, \dot{P}, \dot{\sigma}) = \overline{\Delta}(\sigma, P^{-1}\dot{P}, \dot{\sigma}) = \begin{cases} \sum_{j=1}^{N_s} \tau^j \dot{\sigma}_j, & \text{if (2.9) holds,} \\ \infty, & \text{otherwise.} \end{cases} \tag{2.10}$$

Let us specialize the foregoing considerations even more and focus on the case where only one slip system is active. For literature dealing with modeling aspects of single-slip systems we refer the reader to [24, 14, 42]. Note that we are still working within the regime of no hardening. Let $s, m \in \mathbb{S}^{n-1}$ with $s \perp m$ determine the single slip system and let $\tau > 0$ stand for the threshold value of the resolved shear stress. Here again $\sigma$ denotes the slip strains along $(s, m)$. We also postulate the common initial conditions $u(0) = u^0$, $P(0) = \mathbb{I}$ and $\sigma(0) = 0$ with given $u^0 : \Omega \to \mathbb{R}^n$. First we follow the approach via dissipation distances. The ideas based on the flow rule will be address in the next paragraph.
Accounting for the two opposite orientations of the slip system one has, strictly speaking, that $N_s = 2$, $s^1 \otimes m^1 = -s^2 \otimes m^2 = s \otimes m$, $\tau^1 = \tau^2 = \tau$ and $\sigma = (\sigma_1, \sigma_2)$ with $\dot{\sigma}_1, \dot{\sigma}_2 \geq 0$. With these assumptions in mind the flow rule (2.9) can be rewritten as

$$P = \mathbb{I} + (\sigma_1 - \sigma_2) s \otimes m = \mathbb{I} - \gamma s \otimes m, \tag{2.11}$$

using the definition $\gamma = \sigma_2 - \sigma_1$. Indeed, $\dot{P} = P(\dot{\sigma}_1 - \dot{\sigma}_2)(s \otimes m) = -\dot{\gamma}(Ps) \otimes m$ yields $\dot{P}s = 0$. In combination with the initial condition $P(0) = \mathbb{I}$ this leads to $Ps = s$ and hence $\dot{P} = -\dot{\gamma} s \otimes m$. Regarding $\gamma(0) = 0$ and $P(0) = \mathbb{I}$ integration finally implies (2.11). Obviously, $\det P = 1$, so that volume conservation is guaranteed for the plastic part of the deformation.
Then, (2.10) provides the adapted formula for the dissipation potential

$$\Delta(P, \sigma, \dot{P}, \dot{\sigma}) = \begin{cases} \tau(\dot{\sigma}_1 + \dot{\sigma}_2), & \text{if } P = \mathbb{I} + (\sigma_1 - \sigma_2) s \otimes m, \\ \infty, & \text{otherwise.} \end{cases} \tag{2.12}$$

We assume that a time-discretization can be picked such that there is plastic strain in at most one of these two directions per time increment. So at least one of the components of $\sigma$ remains constant in every single step. This can be achieved by choosing a monotone loading path. If you want to analyze for example cyclic loading, matters are more subtle. Here we point to [44] for recent developments in this context. In view of this additional requirement we infer from (2.12) and the definition of the dissipation distance that

$$D(\hat{P},\hat{\sigma},\widetilde{P},\tilde{\sigma}) = \hat{D}(\hat{P},\hat{\gamma},\widetilde{P},\tilde{\gamma}) = \begin{cases} \tau|\hat{\gamma}-\tilde{\gamma}|, & \text{if } \hat{P} = \mathbb{I} - \hat{\gamma} s\otimes m,\ \widetilde{P} = \mathbb{I} - \tilde{\gamma} s\otimes m, \\ \infty, & \text{otherwise.} \end{cases}$$

The above formula reveals that the internal variable $P$ can be essentially described by the single parameter $\gamma$ and vice versa, so that one of these quantities, here $P$, can be eliminated. After locally minimizing out $\gamma$, the condensed energy of the system is finally given by

$$\hat{\psi}_{\text{cond}}(\tilde{\gamma};F) = \inf_{\gamma\in\mathbb{R}} \left\{ \overline{\psi}(F(\mathbb{I}-\gamma s\otimes m)) + \tau\,|\gamma-\tilde{\gamma}| \right\}$$

with $\tilde{\gamma}\in\mathbb{R}$ and $F\in\mathbb{R}^{n\times n}$. In analogy to (2.8) one can now formulate the associated incremental problem.

### 2.3.2. Incorporating isotropic hardening

In order to include isotropic hardening into the previous single-slip model we use the concept based on the flow rule and the yield function and follow [14]. By the way, comparison with Section 2.3.1 will show that both approaches lead to the same incremental problem in absence of hardening.
Let $\overline{\psi}(FP,p)$ be the stored-energy density, $Q$ and $q$ the dual variables connected with $P$ and the scalar hardening parameter $p\in\mathbb{R}$, respectively, $\overline{Q} = P^TQ$, $(s,m)$ the active slip system and $\tau$ the yield stress. Then the yield function is given by

$$\overline{\varphi}(\overline{Q},q) = |s\cdot\overline{Q}m| - \tau - q.$$

Using formula (2.2), we obtain the flow rule

$$(P^{-1}\dot{P},\dot{p}) = \dot{\sigma}\big(\text{sign}\,(s\cdot\overline{Q}m)s\otimes m, -1\big) = \big(-\dot{\gamma}s\otimes m, -|\dot{\gamma}|\big)$$

for $\overline{\varphi}\leq 0\leq\dot{\sigma}$ with $\dot{\sigma}\overline{\varphi}=0$, where the consistency parameter $\dot{\sigma}\geq 0$ can be interpreted as the slip-rate of the system and $\gamma:\mathbb{R}\to\mathbb{R}$ is defined such that $\dot{\gamma} = -\dot{\sigma}\,\text{sign}\,(s\cdot\overline{Q}m)$ and $\gamma(0)=0$. In view of the initial condition $P^0=\mathbb{I}$, we derive

$$P = \mathbb{I} - \gamma s\otimes m \tag{2.13}$$

similarly to the calculations in the paragraph before. Here as well $P$ can be expressed in terms of $\gamma$. With $\overline{\varphi}$ known, one explicitly computes the dissipation function as

$$\chi^*(P,p) = \hat{\chi}^*(\gamma,p) = \begin{cases} \tau|\gamma|, & \text{if } |\gamma|+p\leq 0\,, \\ \infty, & \text{otherwise,} \end{cases} \tag{2.14}$$

see [14, Chapter 6] for the details of the argumentation. Inserting the relation (2.13) and the dissipation function $\chi^*$ of (2.14) into functional (2.4) provides for $k \in \{1, \dots, N\}$,

$$E^k[u, \gamma, p] = \int_\Omega \overline{\psi}(\nabla u(\mathbb{I} - \gamma s \otimes m), p) + \tau|\gamma - \gamma^{k-1}| \,\mathrm{d}x - \langle l(t^k), u\rangle, \quad (2.15)$$

if $|\gamma - \gamma^{k-1}| + p - p^{k-1} \leq 0$ and $E^k[u, \gamma, p] = \infty$, otherwise.
In the following we suppose the stored energy density $\overline{\psi}$ consists of a purely elastic component $\psi_{\mathrm{el}}$ depending only on $F_{\mathrm{el}}$ and an axially symmetric hardening energy density $\psi_{\mathrm{h}}$, in formulas $\overline{\psi}(F_{\mathrm{el}}, p) = \psi_{\mathrm{el}}(F_{\mathrm{el}}) + \psi_{\mathrm{h}}(p)$. Taking linear hardening, for instance, one obtains $\psi_{\mathrm{h}}(p) = \frac{1}{2}ap^2$, where $a > 0$ is the hardening modulus, while $\psi_{\mathrm{h}} \equiv 0$ in a model neglecting hardening effects.
Eliminating the internal parameters $p$ and $\gamma$ in the integrand of (2.15) by pointwise minimization one finds the reduced energy density

$$\hat{\psi}_{\mathrm{cond}}(\tilde{\gamma}, \tilde{p}; F) = \inf_{\gamma \in \mathbb{R}} \left\{ \psi_{\mathrm{el}}\big(F(\mathbb{I} - \gamma s \otimes m)\big) + \psi_{\mathrm{h}}(\tilde{p} - |\gamma - \tilde{\gamma}|) + \tau|\gamma - \tilde{\gamma}| \right\}.$$

If we focus our attention on the first time step and assume the initial conditions $p^0 = p(0) = 0$ and $P(0) = \mathbb{I}$, which implies $\gamma^0 = \gamma(0) = 0$, we have to solve:

Find $u : \Omega \to \mathbb{R}^n$ with $u = u_b$ on $\partial\Omega$ which minimizes

$$E_{\mathrm{cond}}[u] = \int_\Omega \psi_{\mathrm{cond}}(\nabla u) \,\mathrm{d}x - \langle l, u\rangle, \quad (2.16)$$

where $l$ is the external loading in the first time step and

$$\psi_{\mathrm{cond}}(F) = \hat{\psi}_{\mathrm{cond}}(0, 0; F) = \inf_{\gamma \in \mathbb{R}} \left\{ \psi_{\mathrm{el}}\big(F(\mathbb{I} - \gamma s \otimes m)\big) + \psi_{\mathrm{h}}(|\gamma|) + \tau|\gamma| \right\}$$

the condensed energy density.

Notice that all the results of this thesis are set within the framework of the first incremental problem. This is quite a strong simplification, of course. Even if Conti and Theil [24] managed to determine explicit solutions for finitely many time-steps in the concrete example of a simple-shear test, considering several incremental problems and hence accounting for the deformation history of the material makes the relevant energies more complicated. Hence, the problem of time evolution is in general difficult to access by analytical methods. Some more comments on this issue can be found in Chapter 9.

# 3. Mathematical concepts

This section is meant to give a brief overview of the basic mathematical theories being both foundation and motivation for the subsequent observations and arguments.
By now it is a well established fact that lacking quasiconvexity of energy densities, such as the integrand $\psi_{\text{cond}}$ of $E_{\text{cond}}$ in (2.16), prevents the associated energy functionals from having minimizers and hence leads to the formation of microstructure, see [7, 8]. First we want to say a word on this fundamental weaker notion of convexity and state its most important features.
Apart from studying fine-scale structures in itself it is of great importance to understand their impact on the material behavior at a macroscopic level. That is where the mathematical technique called relaxation, which relies on the notion of quasiconvexity, comes in. Relaxation basically consists in minimizing the system energy with respect to all possible microstructures and thus helps to understand the macroscopic properties without caring about unnecessary details caused by microscopic effects.
Large parts of this thesis are based on the method of $\Gamma$-convergence. This is primarily a concept of convergence for variational problems involving parameters. In particular, it provides effective problems which characterize the behavior of minimizers or low energy states as the parameters tend to their limit.

## 3.1. Notions of convexity

Based on [25, 19, 62, 30], we want to discuss the idea of quasiconvexity, which was introduced originally by Morrey [59] in 1952. In the following we use the notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and present a generalized version of Morrey's definition for extended-valued functions.

**Definition 3.1 (Quasiconvexity)** *Suppose $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}}$. Then, $f$ is called quasiconvex, if it holds for every bounded, open and nonempty set $\Omega \subset \mathbb{R}^n$ with $|\partial\Omega| = 0$ that*

$$f(F) \leq \frac{1}{|\Omega|} \int_\Omega f(F + \nabla\varphi)\,\mathrm{d}x \qquad \text{for all } F \in \mathbb{R}^{m\times n} \text{ and } \varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^m), \tag{3.1}$$

*whenever the integral on the right-hand side exists.*

**Remark 3.2** *It can be shown that* (3.1) *is true for all the sets $\Omega$, as soon as it holds for one of them, see [19, Corollary 1.6] for a proof via Vitali's covering theorem.*

Notice that quasiconvexity is the natural notion of convexity in connection with vector-valued variational problems involving integral functionals. Indeed, it is closely linked to the existence of minimizers, as the next theorem shows.

**Theorem 3.3** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $f : \mathbb{R}^{m\times n} \to \mathbb{R}$. We consider maps $u : \Omega \to \mathbb{R}^m$ and define the functional*

$$I[u] = \int_\Omega f(\nabla u)\,\mathrm{d}x.$$

*Suppose that $f$ fulfills the growth and coercivity condition*

$$c\,|F|^p \le f(F) \le C\,(|F|^p + 1) \qquad \text{for all } F \in \mathbb{R}^{m\times n}, \tag{3.2}$$

*with $p \in (1,\infty)$ and constants $c, C > 0$. Then the following statements hold true.*

*(i) If $f$ is quasiconvex, $I$ is weakly lower semi-continuous in $W^{1,p}(\Omega;\mathbb{R}^m)$, i.e.*

$$I[u] \le \liminf_{k\to\infty} I[u^k]$$

*for any sequence $\{u^k\}_{k\in\mathbb{N}}$ with $u^k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$.*

*(ii) If $f$ is quasiconvex and $v \in W^{1,p}(\Omega;\mathbb{R}^m)$ is given, there exists a minimizer of $I$ in the class $W^{1,p}_v(\Omega;\mathbb{R}^m) := \{u \in W^{1,p}(\Omega;\mathbb{R}^m) \mid u - v \in W^{1,p}_0(\Omega;\mathbb{R}^m)\}$.*

The proof of (ii) is classically done with the direct method in the calculus of variations. It uses essentially that quasiconvexity of $f$ implies weak lower semi-continuity of $I$, as stated in (i). More details can be found in the references quoted at the beginning of this section.
Since quasiconvexity is in general hard to verify for concrete functions, it is helpful to look out for necessary and sufficient conditions that are easier to check. So let us define two other notions of convexity, which go back to Ball [9] and turn out to fulfill exactly these requirements. First we formulate the slightly stronger property named polyconvexity.

**Definition 3.4 (Polyconvexity)** *A function $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}}$ is said to be polyconvex, if there exists a convex function $g : \mathbb{R}^{\tau(m,n)} \to \overline{\mathbb{R}}$ such that*

$$f(F) = g\big(M(F)\big) \qquad \text{for all } \ F \in \mathbb{R}^{m\times n},$$

*where $M(F)$ is the vector of all minors, i.e. subdeterminants of $F$, and $\tau(m,n)$ denotes the length of $M(F)$.*
*Further, a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called polyaffine, if $f$ and $-f$ are polyconvex.*

**Remark 3.5** *1. In this work we are merely interested in the cases* $n = m = 2$ *and* $m = n = 3$*. If* $n = m = 2$*,* $M$ *takes the form* $M(F) = (F, \det F) \in \mathbb{R}^5$ *for all* $F \in \mathbb{R}^{2\times 2}$*, while considering* $m = n = 3$ *yields* $M(F) = (F, \operatorname{cof} F, \det F) \in \mathbb{R}^{19}$ *for all* $F \in \mathbb{R}^{3\times 3}$*.*
*2. The reference [25, Section 4.1.2.1, Theorem 1.5] provides a useful characterization of polyaffine functions, which in the case* $m = n = 2$ *reads as follows. Any polyaffine function* $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ *is of the form*

$$f(F) = A : F + b \det F + c \tag{3.3}$$

*with* $A \in \mathbb{R}^{2\times 2}$ *and* $b, c \in \mathbb{R}$*.*

Second, we present a type of convexity which is weaker in comparison to quasiconvexity.

**Definition 3.6 (Rank-one convexity)** *A function* $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}}$ *is rank-one convex, if*

$$f\big(\lambda F + (1-\lambda)G\big) \;\le\; \lambda f(F) + (1-\lambda) f(G)$$

*for every* $\lambda \in [0,1]$ *and all* $F, G \in \mathbb{R}^{m\times n}$ *with* $\operatorname{rank}(F - G) = 1$*. Equivalently, we can say that* $f$ *needs to be convex along all rank-one lines, meaning that the mappings* $t \to f(F + tR)$ *are convex for all* $F \in \mathbb{R}^{m\times n}$ *and* $R \in \mathbb{R}^{m\times n}$ *with* $\operatorname{rank}(R) = 1$*.*

In short one can summarize the established relations between the different notions of convexity as follows: If $f$ is finite-valued, one has

$$f \text{ convex} \;\Rightarrow\; f \text{ polyconvex} \;\Rightarrow\; f \text{ quasiconvex} \;\Rightarrow\; f \text{ rank-one convex}, \tag{3.4}$$

whereas it holds for $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}}$ that

$$f \text{ convex} \;\Rightarrow\; f \text{ polyconvex} \;\Rightarrow\; f \text{ rank-one convex} \tag{3.5}$$

and

$$f \text{ convex} \;\Rightarrow\; f \text{ polyconvex} \;\Rightarrow\; f \text{ quasiconvex}. \tag{3.6}$$

The implications (3.4)-(3.6) are by now standard results, so we dispense with repeating the proofs here again and refer to either of [25, 19, 62]. Additionally, it was shown that the reverse implications are not true in general, so that these notions are really distinct. The question whether rank-one convexity yields quasiconvexity is the most difficult to answer. For $m \ge 3$ there is this famous counterexample by Šverák [73], but the case $m = 2$ has not been solved yet.
Notice that all the different notions of convexity are equivalent in the scalar case, that is, if $n = 1$ or $m = 1$.

## 3.2. Relaxation

When speaking of relaxation from a physical viewpoint, one essentially means the transition from microscopic to macroscopic energies. This is studying large scale effects by taking the average over all possible microstructures. The present paragraph is intended to motivate, concretize and justify this approach by mathematical means.

### Mathematical definition

We start by giving the mathematical definition of relaxation in a rather general context following [26, Chapter 3]. There the interested reader can find the proofs of the stated results as well as further details on the topic.

**Definition 3.7** *Let $(X, d)$ be a metric space and $f : X \to \overline{\mathbb{R}}$. The lower semicontinuous envelope $sc^- f$ of $f$, which is also named the relaxed function of $f$, is defined for all $x \in X$ as*

$$(sc^- f)(x) = \sup\big\{g(x) \mid g : X \to \overline{\mathbb{R}} \ \textit{ lower semicontinuous } \textit{ with } g \leq f\big\}.$$

*Recall that a function $g : X \to \overline{\mathbb{R}}$ is lower semicontinuous, if $g(x) \leq \liminf_{k\to\infty} g(x^k)$ for all sequences $\{x^k\}_{k\in\mathbb{N}} \subset X$ with $x^k \to x$ in $X$.*

There is an alternative and useful characterization of $sc^- f$.

**Proposition 3.8** *If $f : X \to \overline{\mathbb{R}}$ and $x \in X$, then $(sc^- f)(x)$ is determined by these two properties:*

*(i) Lower bound: For every sequence $\{x^k\}_{k\in\mathbb{N}}$ converging to $x$ in $X$, one has*

$$(sc^- f)(x) \leq \liminf_{k\to\infty} f(x^k).$$

*(ii) Recovery sequence: There exists a sequence $\{x^k\}_{k\in\mathbb{N}}$ converging to $x$ in $X$, such that*

$$(sc^- f)(x) = \lim_{k\to\infty} f(x^k).$$

Now we focus on minimum problems, the main issue in calculus of variations. The next theorem shows a close relation between $\inf_{x\in X} f(x)$ and its relaxed problem connecting minimizers of $sc^- f$ to minimizing sequences of $f$.

**Theorem 3.9** *Suppose $f : X \to \overline{\mathbb{R}}$ is coercive, meaning that the sets $\{f \leq t\}$ are compact for all $t \in \mathbb{R}$. Then $sc^- f$ attains its infimum in $X$ and it holds*

$$\inf_{x\in X} f(x) = \min_{x\in X}(sc^- f)(x).$$

*Moreover, the cluster points of minimizing sequences of $f$ in $X$ coincide with the minimizers of $sc^- F$ in $X$.*

By passing from $\inf_{x \in X} f(x)$ to its relaxed counterpart one replaces a variational problem which may not be solvable by one which has a minimizer. Besides, this minimizer constitutes the limit of a minimizing sequence of $f$, which makes relaxation a powerful tool to study of the original problem.
In applications, however, $X$ is often a function space endowed with the weak topology so that specific features of the minimum points of $sc^- f$ may fail to reappear in the low energy states of $f$. Instead the minimizers of the relaxed problem rather characterize the average over infimizing elements of $f$. In the paragraph after the next the reader finds a more concrete discussion on this matter.

### Quasiconvex envelopes and hulls

As we will see later on quasiconvexity plays a crucial role regarding the relaxation of integral functionals. Actually, investigating quasiconvexifications of the usually non-quasiconvex integrands has turned out to provide new fundamental insight.
First there are some definitions to be made. They, as well as most of this section, are taken from [19, 62, 25].

**Definition 3.10** *For given $f : \mathbb{R}^{m \times n} \to \overline{\mathbb{R}}$ the quasiconvex envelope $f^{\mathrm{qc}}$ is defined to be the largest quasiconvex function smaller or equal to $f$. In formulas,*

$$f^{\mathrm{qc}}(F) = \sup\big\{g(F) \mid g : \mathbb{R}^{m \times n} \to \overline{\mathbb{R}} \;\; \textit{quasiconvex with } g \leq f\big\} \tag{3.7}$$

*for all $F \in \mathbb{R}^{m \times n}$.*
*The polyconvex and the rank-one convex envelopes of $f$ are defined similarly and denoted by $f^{\mathrm{pc}}$ and $f^{\mathrm{rc}}$, respectively.*

**Remark 3.11** *1. These definitions are in accordance with the definition of the convex envelope $f^{\mathrm{c}}$ of $f$, which is the largest convex lower bound on $f$.*
*2. Concerning the question of existence of the envelopes defined above there is a positive answer, for which we refer for example to [19, Lemma 3.2]. In fact, the supremum in* (3.7) *is a maximum unless $f^{\mathrm{qc}} \equiv -\infty$.*
*3. The polyconvex envelope of a function $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ can be rewritten with the help of polyaffine functions as*

$$f^{\mathrm{pc}}(F) = \sup\big\{g(F) \mid g : \mathbb{R}^{m \times n} \to \mathbb{R} \;\; \textit{polyaffine with } g \leq f\big\}, \tag{3.8}$$

*see [25, Section 5.1.1.2].*

By (3.4) we conclude for finite-valued functions $f$ that

$$f^{\mathrm{pc}} \leq f^{\mathrm{qc}} \leq f^{\mathrm{rc}}, \tag{3.9}$$

while in view of (3.5) the inequality $f^{\mathrm{qc}} \leq f^{\mathrm{rc}}$ is in general not true for extended-valued $f$.
For quasiconvex envelopes of finite-valued functions there exists the following instructive representation.

**Proposition 3.12** *The quasiconvex envelope of* $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ *is given by*

$$f^{\mathrm{qc}}(F) = \inf_{\varphi \in W_0^{1,\infty}((0,1)^n;\mathbb{R}^m)} \int_{(0,1)^n} f\big(F + \nabla\varphi(x)\big)\, \mathrm{d}x. \tag{3.10}$$

This formula points out that the quasiconvex envelope of $f$ in $F \in \mathbb{R}^{m\times n}$ is calculated by minimizing over all Lipschitz functions $\varphi$ satisfying the affine boundary condition $\varphi(x) = Fx$ on $\partial(0,1)^n$.

**Remark 3.13** *In case that $f$ is allowed to be extended-valued one can so far only prove that the right-hand side in* (3.10) *is an upper bound on the quasiconvexification of $f$, see the proofs of [19, Lemma 3.2, Proposition 3.3]. The reverse inequality is still an open problem.*

For future purpose let us reformulate characterization (3.10) of $f^{\mathrm{qc}}$ in terms of measures, as it is done e.g. in [10, Chapter 2]. Suppose $\mathcal{PM}(\mathbb{R}^{m\times n}) \subset \mathcal{M}(\mathbb{R}^{m\times n})$ denotes the space of probability measures on $\mathbb{R}^{m\times n}$, where $\mathcal{M}(\mathbb{R}^{m\times n}) \cong C_0^0(\mathbb{R}^{m\times n})'$ is the space of signed, regular measures on $\mathbb{R}^{m\times n}$ and $C_0^0(\mathbb{R}^{m\times n})$ the space of continuous real-valued functions which converge to zero at infinity. The corresponding duality pairing reads $\langle \nu, g\rangle_{\mathcal{M}(\mathbb{R}^{m\times n}), C_0^0(\mathbb{R}^{m\times n})} = \int_{\mathbb{R}^{m\times n}} g \,\mathrm{d}\nu$. Further let $\mathcal{P}$ be the subset of $\mathcal{PM}(\mathbb{R}^{m\times n})$ containing all $\nu$ represented by

$$\langle \nu, h\rangle = \int_{(0,1)^n} h(F + \nabla\varphi)\,\mathrm{d}x \qquad \text{for all } h \in C_0^0(\mathbb{R}^{m\times n}),$$

with some $F \in \mathbb{R}^{m\times n}$ and $\varphi \in W_0^{1,\infty}((0,1)^n;\mathbb{R}^m)$. Then,

$$f^{\mathrm{qc}}(F) = \inf_{\nu\in\mathcal{P},\, \overline{\nu}=F} \int_{\mathbb{R}^{m\times n}} f \,\mathrm{d}\nu, \tag{3.11}$$

where $\overline{\nu} = \int_{\mathbb{R}^{m\times n}} \mathrm{id}\,\mathrm{d}\nu$ stands for the center of mass of $\nu$ and id is the identical mapping on $\mathbb{R}^{m\times n}$.
Next we extend the different notions of convexity from functions to sets. First let us recall the definition of a convex set and the convex hull of a set.

**Definition 3.14** *A set $\Sigma \subset \mathbb{R}^{m\times n}$ is called convex, if $\lambda F + (1-\lambda)G \in \Sigma$ for all $\lambda \in [0,1]$ and all $F, G \in \Sigma$.*
*If the set $\Sigma$ is not convex, the smallest convex set containing $\Sigma$ is called its convex envelope and is denoted by $\Sigma^{\mathrm{c}}$.*

An alternative way of defining $\Sigma^{\mathrm{c}}$ is via separation, meaning that $\Sigma^{\mathrm{c}}$ is exactly the set of all points that cannot be separated from $\Sigma$ by convex functions. In formulas,

$$\Sigma^{\mathrm{c}} = \left\{ F \in \mathbb{R}^{m\times n} \mid f(F) \leq \sup_{G\in\Sigma} f(G) \text{ for all } f : \mathbb{R}^{m\times n} \to \mathbb{R} \text{ convex} \right\}.$$

This definition can be generalized to the other notions of convexity.

**Definition 3.15** *Let $\Sigma \subset \mathbb{R}^{m\times n}$. Then the set*

$$\Sigma^{\mathrm{qc}} = \left\{ F \in \mathbb{R}^{m\times n} \mid f(F) \leq \sup_{G\in\Sigma} f(G) \textit{ for all } f : \mathbb{R}^{m\times n} \to \mathbb{R} \textit{ quasiconvex} \right\}.$$

*is called the quasiconvex hull of $\Sigma$.*
*The polyconvex hull $\Sigma^{\mathrm{pc}}$ and the rank-one convex hull $\Sigma^{\mathrm{rc}}$ are defined analogously.*
*Further, a set $\Sigma \subset \mathbb{R}^{m\times n}$ is said to be quasiconvex, if $\Sigma = \Sigma^{\mathrm{qc}}$. Likewise we define rank-one convex and polyconvex sets.*

Besides, one can define the lamination convex hull, which is based on the notion of rank-one convexity.

**Definition 3.16** *For $\Sigma \subset \mathbb{R}^{m\times n}$ let $\Sigma^{(0)} = \Sigma$ and define inductively for $j \in \mathbb{N}$,*

$$\Sigma^{(j)} = \Sigma^{(j-1)} \cup \left\{ \lambda F + (1-\lambda)G \mid F, G \in \Sigma^{(j-1)} \textit{ with } \operatorname{rank}(F-G) = 1,\ \lambda \in (0,1) \right\}.$$

*Then, $\Sigma^{\mathrm{lc}} = \bigcup_{j=1}^{\infty} \Sigma^{(j)}$ is called the lamination convex hull of $\Sigma$.*

As a consequence of (3.4) there are the inclusions

$$\Sigma^{\mathrm{rc}} \subset \Sigma^{\mathrm{qc}} \subset \Sigma^{\mathrm{pc}} \subset \Sigma^{\mathrm{c}}.$$

For the verification of the inclusion $\Sigma^{\mathrm{lc}} \subset \Sigma^{\mathrm{rc}}$, as well as for an example of a set with trivial lamination convex hull but nontrivial rank-one convex hull, we point to [19, Chapter 4].

### A classical relaxation theorem for integral functionals

Next let us focus on variational problems involving integral functionals. In the first instance we will be concerned with hightlighting the connection between relaxation as introduced at the beginning of this chapter and the quasiconvex envelopes of the associated integrands.
For a given function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ we define the functional

$$I[u] = \int_\Omega f(\nabla u)\,\mathrm{d}x, \tag{3.12}$$

where $u : \Omega \to \mathbb{R}^m$ and $\Omega \subset \mathbb{R}^n$ an open and bounded set. Functionals of this shape naturally occur in nonlinear elasticity, but also in finite plasticity describing the energy of one incremental step, compare (2.16). Actually, the integrand $f$ is mostly not quasiconvex.
In this setting the classical relaxation result, which is quoted frequently in the literature, e.g. [25, 19, 62] and the references therein, reads as follows.

**Theorem 3.17** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfy growth condition* (3.2) *for $p \in (1, \infty)$. We define*

$$\bar{I}[u] = \int_\Omega f^{\mathrm{qc}}(\nabla u)\, \mathrm{d}x$$

*and take $W^{1,p}_v(\Omega;\mathbb{R}^m)$ as in the statement of Theorem 3.3 with $v \in W^{1,p}(\Omega;\mathbb{R}^m)$ given. Then,*

$$\inf_{u\in W^{1,p}_v(\Omega;\mathbb{R}^m)} I[u] = \min_{u\in W^{1,p}_v(\Omega;\mathbb{R}^m)} \bar{I}[u].$$

*Besides, for all $u \in W^{1,p}_v(\Omega;\mathbb{R}^m)$ there exists a sequence $\{u^k\}_{k\in\mathbb{N}} \subset W^{1,p}_v(\Omega;\mathbb{R}^m)$ such that $u^k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and $I[u^k] \to \bar{I}[u]$ as $k \to \infty$.*

In particular, every minimizer of $\bar{I}$ is an accumulation point (in the sense of weak-$W^{1,p}(\Omega;\mathbb{R}^m)$ convergence) of a minimizing sequence for $I$. As a consequence of the preceding theorem and the weak lower semicontinuity of $\bar{I}$ by Theorem 3.3 (i) we observe that $\bar{I}$ is exactly the relaxation of $I$ in the sense of Proposition 3.8.
If we assume $I$ to be an energy functional modeling material behavior, the features of relaxation imply an entire characterization of the macroscopic effects by the good-natured functional $\bar{I}$. More precisely, the minimizing sequences of $I$ symbolize microstructure, while the minimizers of $\bar{I}$ cover average properties and hence represent material response on a larger scale. In view of Theorem 3.17 it is not surprising that the structure of the minimizing sequences of $I$ may not carry over to their limit, since one has convergence merely in the weak sense, which justifies the idea of $\bar{I}$ capturing averaged material behavior. See [23, Section 3 and 4] for further physical motivation on the whole subject.

**Remark 3.18** *All the results naturally extend to functionals with continuous perturbations, which speaking in terms of applications occur as body forces or surface traction. To be exact, the relaxation of $I[u] + T(u)$ is given by $\bar{I}[u] + T(u)$ provided $T(u)$ is such a continuous perturbation. A rigorous mathematical argument for this assertion can be found in [26]. Notice that relaxation can be interpreted as a special case of $\Gamma$-convergence, which is stable under addition of continuous functions, and compare Section 3.3.*

### The physical notion of relaxation

The previous observations motivate and suggest the definition of relaxation as it is commonly used in physics and materials science. Here the functional

$$\bar{I}[u] = \int_\Omega f^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x \tag{3.13}$$

is called the relaxation (or relaxed functional) of $I$ as in (3.12). The idea behind is that in view of the representation formula of $f^{\mathrm{qc}}$ in Proposition 3.12 macroscopic behavior can be described by locally minimizing over possible fine-scale structures. Here we want to point out that this notion is applied no matter whether the requirements of Theorem 3.17 are strictly fulfilled.
For the rest of this thesis we will refer to relaxation in this new sense and replace energy functionals by their relaxed counterparts in order to describe large-scale effects.
Concluding, let us remark that there are physically relevant situations in which the classical relaxation of Definition 3.7 does not coincide with the expression (3.13). For a concrete example see [23], where Conti and Ortiz study a model of single-crystal plasticity with infinite latent hardening in the framework of linearized kinematics.

## 3.3. Γ-convergence

Many mathematical problems originating from applications contain parameters. In particular, this is the case for the single-slip model with elastic energy which is at the heart of this thesis. Conceptually, one has to treat a family of variational problems of the form

$$\min\{f_\varepsilon(x) \mid x \in X\}, \tag{3.14}$$

where $\varepsilon > 0$ is the mentioned parameter and $X$ a metric space. Our goal is to understand the system for small $\varepsilon$ by studying its limit behavior as $\varepsilon$ tends to zero. This can be done replacing the family of (3.14) by an effective problem

$$\min\{f(x) \mid x \in X\}. \tag{3.15}$$

The latter is to be independent of the parameter, but should capture the relevant properties of minimizers of (3.14) for small $\varepsilon$. Besides, it is desirable if solutions of (3.15) are easier to achieve than those of the original problem. De Giorgi and Franzoni [27] were the first to come up with a variational convergence suitable for realizing such an approach. This paragraph, which relies on [11, 26, 2], is intended to make the reader familiar with the fundamental aspects of Γ-convergence.
According to the introductory lines of this chapter Γ-convergence is a notion mainly used in the context of integral problems. Nevertheless we keep its definition and the presentation of the corresponding results rather general.

**Definition 3.19** *Let $X$ be a metric space with metric $d$, $f_\varepsilon : X \to \overline{\mathbb{R}}$ for $\varepsilon > 0$ and $f : X \to \overline{\mathbb{R}}$. The sequence $\{f_\varepsilon\}_{\varepsilon>0}$ is said to $\Gamma$-converge to $f$ as $\varepsilon \to 0$, in symbols $f_\varepsilon \xrightarrow{\Gamma} f$, if the following two properties are fulfilled for all $x \in X$:*

*(i) Lower bound: For every sequence $\{x_\varepsilon\}_{\varepsilon>0}$ converging to $x$ in $X$, it holds*

$$f(x) \leq \liminf_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon).$$

*(ii) Recovery sequence: There is a sequence $\{x_\varepsilon\}_{\varepsilon>0}$ converging to $x$, such that*

$$f(x) = \lim_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon).$$

*Then, $f$ is called the $\Gamma$-limit of $f_\varepsilon$, in symbols $f = \Gamma\text{-}\lim_{\varepsilon\to 0} f_\varepsilon$.*

**Remark 3.20** *Comparing Definition 3.19 to the characterization of $sc^- f$ in Proposition 3.8 shows that mathematical relaxation can be interpreted as a special $\Gamma$-limit. Indeed, one simply has to substitute $\{f_\varepsilon\}_\varepsilon$ by the constant sequence $\{f\}$.*

The next statement provides a summary of the main properties of $\Gamma$-convergence. It is only meant to give a brief overview. All the material presented here is discussed at length, including proofs, in the literature quoted above.

With the notation as in Definition 3.19 we find:

(LS) *Lower semicontinuity of $\Gamma$-limits:* Any $\Gamma$-limit $f$ is lower semicontinuous in X.

(S) *Stability under continuous perturbations:* If $g : X \to \mathbb{R}$ is continuous and $f_\varepsilon \xrightarrow{\Gamma} f$, then $f_\varepsilon + g \xrightarrow{\Gamma} f + g$.

(M) *Convergence of minima:* Suppose that $\overline{x}_\varepsilon \in X$ are minimizers of $f_\varepsilon$ for $\varepsilon > 0$. Then every cluster point of $\{\overline{x}_\varepsilon\}_{\varepsilon>0}$ minimizes $f$ in $X$.

Let us emphasize that every $\Gamma$-convergence result should be coupled to a compactness result of the form:

(C) *Compactness:* Any bounded energy sequence $\{x_\varepsilon\}_{\varepsilon>0}$, i.e. a sequence $\{x_\varepsilon\}_{\varepsilon>0}$ satisfying $f_\varepsilon(x_\varepsilon) \leq C$ for some constant $C > 0$, is relatively compact in $X$.

This way sequences $\{\overline{x}_\varepsilon\}_{\varepsilon>0}$ of minimizers of $f_\varepsilon$ are ensured to be a priori relatively compact, meaning that they possess subsequences converging in $X$ and hence cluster points. This prevents property (M) from being an empty statement. In case that $f_\varepsilon$ has no minimizers one instead obtains a similar result for 'almost'-minimizers, the low energy states.

Whenever solving problems with the help of Γ-convergence the choice of the metric $d$ plays an important role. If $X$ is endowed with two different metrics $d$ and $\tilde{d}$, it is generally not clear whether Γ-convergence of a sequence in $(X, d)$ yields Γ-convergence with respect to $(X, \tilde{d})$ and vice versa. If existence of both Γ-limits is assumed, though, one obtains the pointwise estimate

$$\Gamma(d)\text{-}\lim_{\varepsilon\to 0} f_\varepsilon \leq \Gamma(\tilde{d})\text{-}\lim_{\varepsilon\to 0} f_\varepsilon,$$

if the topology induced by the metric $\tilde{d}$ is finer than the one arising from $d$.
A second issue one has to be aware of is that Γ-limits are crucially affected by rescaling. While the sets of minimizers of $f_\varepsilon$ are unchanged under multiplication by scaling factors $\alpha_\varepsilon \in \mathbb{R}$, the Γ-limits of $\{\alpha_\varepsilon f_\varepsilon\}_{\varepsilon>0}$ usually differ for each scaling and carry different information. Therefore one of the tasks before starting to prove a Γ-convergence result is to find the optimal rescaling, which brings about the desired or the largest amount of information.

# 4. Two single-slip models in crystal elastoplasticity

Let us now establish the special types of models which mark the center of this thesis. These are essentially single-slip models as they were discussed earlier. In particular, we consider systems with realistic elastic energies that highly penalize elastic deformations differing from rigid body motions. In the long run we plan on relating the latter to the limiting case of rigid elasticity, which was thoroughly investigated in [24] by analytical methods and is by now well understood. As already suggested, focus will be put on the treatment of the first incremental problem of the time-discretized evolution process.
In this section we pick up on the notation of Chapter 2 and the relevant energy densities derived in Section 2.3.2 and continue within the setting introduced there.

## 4.1. Discussion of a model with rigid elasticity

In contrast to the engineering literature mathematicians commonly denote densities of energy functionals by $W$ instead of $\psi$. Therefore we identify here $\psi_{\mathrm{cond}} = W_{\mathrm{cond}}$, $\psi_{\mathrm{h}} = W_{\mathrm{h}}$ and $\psi_{\mathrm{el}} = W_{\mathrm{el}}$.
As elastically rigid energy density we refer to

$$W_{\mathrm{el}}^{\mathrm{rigid}}(F_{\mathrm{el}}) = \begin{cases} 0 & \text{for } F_{\mathrm{el}} \in \mathrm{SO}(n)\,, \\ \infty & \text{else}\,, \end{cases} \tag{4.1}$$

which says that the admissible elastic deformations are exactly the rigid body rotations with $\mathrm{SO}(n)$ standing for the special orthogonal group. In view of this particular choice of $W_{\mathrm{el}}$, the condensed energy density of (2.16) turns into

$$W_{\mathrm{cond}}(F) = \begin{cases} W_{\mathrm{h}}(|\gamma|) + \tau|\gamma| & \text{for } F \in \mathcal{M}^{(n)}\,, \\ \infty & \text{else}\,, \end{cases} \tag{4.2}$$

where the set $\mathcal{M}^{(n)}$ is defined by

$$\mathcal{M}^{(n)} = \big\{F \in \mathbb{R}^{n\times n} \mid F = R(\mathbb{I} + \gamma s \otimes m),\ R \in \mathrm{SO}(n),\ \gamma \in \mathbb{R}\big\}. \tag{4.3}$$

There are alternative ways of writing $\mathcal{M}^{(n)}$ depending on the space dimension $n \in \{2, 3\}$. Namely,

$$\mathcal{M}^{(2)} = \big\{F \in \mathbb{R}^{2\times 2} \mid \det F = 1, |Fs| = 1\big\}, \tag{4.4}$$

$$\mathcal{M}^{(3)} = \big\{F \in \mathbb{R}^{3\times 3} \mid \det F = 1,\ |(\operatorname{cof} F)(s \wedge m)| = |F(s \wedge m)| = |Fs| = 1\big\}, \tag{4.5}$$

where cof $F$ is the cofactor matrix of $F$, which consists of all $2 \times 2$-minors of $F$. For $s = e_1$ and $m = e_2$ the equalities (4.4) and (4.5) are immediate to derive from (4.3), since every matrix in $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^{3\times 3}$ can be transformed into an upper triangular matrix by premultiplication of a rotation. For arbitrary $s$ and $m$ we use a change of variables argument. The quasiconvex hulls of $\mathcal{M}^{(2)}$ and $\mathcal{M}^{(3)}$, which can be shown to coincide with the polyconvex, rank-one convex and the lamination convex hulls, see Definitions 3.15 and 3.16, were determined in [24] and take the form

$$\begin{aligned} \mathcal{N}^{(2)} &= \big\{F \in \mathbb{R}^{2\times 2} \mid \det F = 1,\ |Fs| \le 1\big\} = (\mathcal{M}^{(2)})^{\mathrm{qc}} = (\mathcal{M}^{(2)})^{\mathrm{rc}}, & (4.6)\\ \mathcal{N}^{(3)} &= \big\{F \in \mathbb{R}^{3\times 3} \mid \det F = 1,\ |(\operatorname{cof} F)(s \wedge m)| = |F(s\wedge m)| = 1,\ |Fs| \le 1\big\} & (4.7)\\ &= \big\{F \in \mathbb{R}^{3\times 3} \mid \det F = 1,\ (\operatorname{cof} F)(s \wedge m) = F(s\wedge m),\ |Fs| \le 1\big\} = (\mathcal{M}^{(3)})^{\mathrm{rc}}. \end{aligned}$$

The equality before the last can be shown under the assumption that $s$ and $m$ are the first two vectors of the standard basis in $\mathbb{R}^3$ by using the formula $\det(F)\mathbb{I} = (\operatorname{cof} F)F^T$. Then again the general case $s, m \in \mathbb{S}^1$ with $s \perp m$ follows again by change of variables. The relevance of $\mathcal{N}^{(2)}$ and $\mathcal{N}^{(3)}$ will become clear later, let us just remark here that these two hulls are important ingredients for the analysis of the energy densities and the associated relaxations we want to study in this work, see Theorem 4.1, Theorem 7.10 and Theorem 8.1.

From now on we intend to restrict to further specializations of (4.2). For $p \ge 1$ we define

$$W(F) = W_p(F) = \begin{cases} |\gamma|^p & \text{for } F \in \mathcal{M}^{(n)}, \\ \infty & \text{else}, \end{cases} \tag{4.8}$$

where we change the notation from $W_{\text{cond}}$ to $W$ (or $W_p$) for sake of simplicity. A possible interpretation of these condensed energy densities is that the case $p = 1$ characterizes a model with dissipation ($\tau = 1$) but no hardening, meaning $W_{\mathrm{h}}(|\gamma|) = 0$, while for $p > 1$ there is actually zero dissipation ($\tau = 0$) and hardening of order $p-1$, i.e. $W_{\mathrm{h}}(|\gamma|) = |\gamma|^p$. The situations without hardening ($p = 1$) and with linear hardening ($p = 2$) are of major interest to us, since they are closely connected with real applications in the field of engineering and materials science, compare [14].

As highlighted in Section 3.2 the key to handle variational problems giving rise to microstructure is the method of relaxation. It is based on the notion of quasiconvexity and provides an appropriate tool for understanding the problem on a macroscopic scale. For the elastically rigid models with energy densities $W_p$ explicit relaxation formulas were calculated in [24], [18] and [21]. These are summarized in the next lemma.

**Theorem 4.1** *Let $W = W_p$ be the energy density as defined in* (4.8) *with $p \in \{1\} \cup [2, \infty)$.*

*If $n = 2$, the poly-, quasi- and rank-one convex envelopes of $W_p$ coincide for each $p \in \{1\} \cup [2, \infty)$ and are given by*

$$W_1^{\mathrm{pc}}(F) = W_1^{\mathrm{qc}}(F) = W_1^{\mathrm{rc}}(F) = \begin{cases} \sqrt{|F|^2 - 2} & \textit{for } F \in \mathcal{N}^{(2)}, \\ \infty & \textit{else}, \end{cases} \tag{4.9}$$

*and in the case $p \geq 2$ by*

$$W_p^{\mathrm{pc}}(F) = W_p^{\mathrm{qc}}(F) = W_p^{\mathrm{rc}}(F) = \begin{cases} (|Fm|^2 - 1)^{p/2} & \text{for } F \in \mathcal{N}^{(2)}, \\ \infty & \text{else} \end{cases} \tag{4.10}$$

*for all $F \in \mathbb{R}^{2\times 2}$, where $\mathcal{N}^{(2)}$ is defined in* (4.6).
*If $n = 3$, the density $W_p$ itself is quasiconvex for all $p \in \{1\} \cup [2, \infty)$. Besides, for $p = 1$ one has*

$$W_1^{\mathrm{pc}}(F) = W_1^{\mathrm{rc}}(F) = \begin{cases} \sqrt{|F|^2 - 3} & \text{for } F \in \mathcal{N}^{(3)}, \\ \infty & \text{else}, \end{cases} \tag{4.11}$$

*while $p \geq 2$ yields*

$$W_p^{\mathrm{pc}}(F) = W_p^{\mathrm{rc}}(F) = \begin{cases} (|Fm|^2 - 1)^{p/2} & \text{for } F \in \mathcal{N}^{(3)}, \\ \infty & \text{else} \end{cases} \tag{4.12}$$

*for all $F \in \mathbb{R}^{3\times 3}$ with $\mathcal{N}^{(3)}$ as in* (4.8).

**Remark 4.2** *In [24] the formulas for the convex envelopes of $W_1$ are given in a different manner using the maximal and minimal nonnegative singular values $\lambda_{\max}(F)$ and $\lambda_{\min}(F)$ of $F \in \mathbb{R}^{n\times n}$ with $n \in \{2, 3\}$. If $n = 2$, there is*

$$W_1^{\mathrm{pc}}(F) = W_1^{\mathrm{qc}}(F) = W_1^{\mathrm{rc}}(F) = \begin{cases} \lambda_{\max}(F) - \lambda_{\min}(F) & \text{for } F \in \mathcal{N}^{(2)}, \\ \infty & \text{else}, \end{cases} \tag{4.13}$$

*and it holds*

$$W_1^{\mathrm{pc}}(F) = W_1^{\mathrm{rc}}(F) = \begin{cases} \lambda_{\max}(F) - \lambda_{\min}(F) & \text{for } F \in \mathcal{N}^{(3)}, \\ \infty & \text{else} \end{cases} \tag{4.14}$$

*in the three-dimensional setting. Hence,* (4.13) *and* (4.14) *provide equivalent formulations to* (4.9) *and* (4.11), *respectively.*

*PROOF of Theorem 4.1.* A detailed proof for $p = 1$ both in two and three space dimensions can be found in [24] and the case $p = 2$ was proven in [18]. Here we present the proof for $p \geq 2$ showing that the same strategy applied for $p = 2$ in [18] carries over. At first let us focus on the 2D setting and determine the polyconvex and the rank-one convex envelopes of the energy density $W_p$ with $p \geq 2$. By $\phi_p$ we denote the right-hand side in the statement of the theorem, that is $\phi_p(F) = \big(|Fm|^2 - 1\big)^{p/2}$ for $F \in \mathcal{N}^{(2)}$ and $\phi_p = \infty$ else. We begin with the essential step of the proof, the construction of an optimal laminate, which will finally provide the desired upper bound on $W_p^{\mathrm{rc}}$. As a motivation for the arguments to come take $F \in \mathcal{N}^{(2)} \setminus \mathcal{M}^{(2)}$ and a rank-one matrix $Y = a \otimes b \in \mathbb{R}^{2\times 2}$ with $a, b \in \mathbb{R}^2 \setminus \{0\}$ satisfying $a^{\perp} F b^{\perp} = 0$. Then we observe that $F + tY \in \mathcal{N}^{(2)}$ for $t \in \mathbb{R}$

provided $|t|$ is sufficiently small and that the function $t \mapsto \phi_p(F + tY)$ is strictly convex unless $Ym = 0$. In view of the determinant constraint $a^\perp F b^\perp = 0$ this is equivalent to $Y = Fm \otimes s$ (up to some real factor). So, from now on let us concentrate on the rank-one line $t \mapsto F_t = F(\mathbb{I} + t\, m \otimes s)$ with $F \in \mathcal{N}^{(2)} \setminus \mathcal{M}^{(2)}$. By $t^+$ and $t^-$ we denote the unique solutions of the quadratic equality

$$1 = |F_t s|^2 = t^2 |Fm|^2 + 2t(Fs \cdot Fm) + |Fs|^2$$

with $t^- < 0 < t^+$. The existence of such $t^{+/-}$ is guaranteed, because $F \in \mathcal{N}^{(2)} \setminus \mathcal{M}^{(2)}$ implies $|Fm| \geq 1$ and $|Fs| < 1$. Define $F^{+/-} = F_{t^{+/-}} \in \mathcal{M}^{(2)}$ and observe $|Fm| = |F^{+/-}m|$. By setting $\lambda = |t^-|/(t^+ - t^-) \in (0,1)$ one obtains $F = \lambda F^+ + (1-\lambda)F^-$. For $G = R(\mathbb{I} + \gamma s \otimes m) \in \mathcal{M}^{(2)}$ with $\gamma \in \mathbb{R}$ and $R \in \mathrm{SO}(2)$ we compute $|Gm|^2 = |m + \gamma s|^2 = 1 + \gamma^2$. Thus, it holds $W_p = \phi_p$ on $\mathcal{M}^{(2)}$. Since $W_p^{\mathrm{rc}}$ is convex along the rank-one line $t \to F_t$, this results in

$$\begin{aligned} W_p^{\mathrm{rc}}(F) &\leq \lambda W_p(F^+) + (1-\lambda) W_p(F^-) \\ &= \lambda\big(|Fm|^2 - 1\big)^{p/2} + (1-\lambda)\big(|Fm|^2 - 1\big)^{p/2} = \phi_p(F), \end{aligned} \tag{4.15}$$

which is the upper bound.

As to the lower bound we find that the function $F \mapsto \max\{|Fm|^2 - 1, 0\}^{p/2}$ is convex and $|Fm| \geq 1$ for all $F \in \mathcal{N}^{(2)}$ by definition. Consequently, the polyconvexity of the set $\mathcal{N}^{(2)}$ leads to polyconvexity of $\phi_p$. If we recall $\phi_p = W_p$ on $\mathcal{M}^{(2)}$ and $W_p = \infty$ on $\mathbb{R}^{2\times 2} \setminus \mathcal{M}^{(2)}$, it is immediate to deduce $\phi_p \leq W_p$ and therefore $\phi_p \leq W_p^{\mathrm{pc}}$. With the estimate $W_p^{\mathrm{pc}} \leq W_p^{\mathrm{rc}}$, see Proposition 3.9, we obtain the postulated statement for $W_p^{\mathrm{pc}}$ and $W_p^{\mathrm{rc}}$.

In order to prove the claim for $W_p^{\mathrm{qc}}$, we can take over the lower bound on $W_p^{\mathrm{pc}}$ arguing with the help of (3.6). Regarding the upper estimate, however, the infinite values of $W_p$ necessitate a different approach, since $W_p^{\mathrm{qc}} \leq W_p^{\mathrm{rc}}$ might not hold in this situation. Here we will apply a version of the convex integration results involving constraints from [24], compare Section 5.7. What we have to prove now is $W_p^{\mathrm{qc}} \leq \phi_p$ on $\mathcal{N}^{(2)} \setminus \mathcal{M}^{(2)}$. To see this, let $F \in \mathcal{N}^{(2)} \setminus \mathcal{M}^{(2)}$ and $k > 0$ such that $|Fm| < k$. Then, by [24, Lemma 2], quoted in Lemma 5.30, there is a function $u \in W^{1,\infty}((0,1)^2; \mathbb{R}^2)$ such that $u(x) = Fx$ on $\partial(0,1)^2$, $\nabla u \in \mathcal{M}^{(2)}$ and $|(\nabla u)m| < k$ almost everywhere. In particular one finds $W_p(\nabla u) = \phi_p(\nabla u) < (k^2 - 1)^{p/2}$ almost everywhere in $(0,1)^2$. Hence, $W_p^{\mathrm{qc}}(F) < (k^2-1)^{p/2}$ by Remark 3.13. Considering the limit $k \to |Fm|$ implies $W_p^{\mathrm{qc}}(F) \leq \phi_p(F)$. In summary we have shown that $W_p^{\mathrm{qc}} \leq \phi_p \leq W_p^{\mathrm{pc}} \leq W_p^{\mathrm{qc}}$. This finishes the proof of the two-dimensional situation.

With respect to the 3D case and the verification of the formula for $W_p^{\mathrm{rc}}$ and $W_p^{\mathrm{pc}}$ it is sufficient to remark that these problems are in principle two-dimensional, so that basically the same construction as above can be applied. For the details of this argumentation we advise the reader to consult [24, Proof of Theorem 2], which extends literally to the case

$p \geq 2$.
In order to show that $W_p$ is quasiconvex in 3D we pursue exactly the lines of [24, Proof of Theorem 2]. There it is proven by a rigidity argument that for given $F \in \mathcal{N}^{(3)}$, any Lipschitz function $u : (0,1)^3 \to \mathbb{R}^3$ satisfying the boundary conditions $u(x) = Fx$ on $\partial(0,1)^3$ and $\nabla u \in \mathcal{M}^{(3)}$ almost everywhere has to be affine. This immediately leads to

$$\int_{(0,1)^3} W_p(\nabla u)\,\mathrm{d}x = W_p(F)$$

for all $u \in W^{1,\infty}((0,1)^3;\mathbb{R}^3)$ with $u(x) = Fx$ on $\partial(0,1)^3$. In view of Definition 3.1 this shows the quasiconvexity of $W_p$ in $F$. $\square$

Let us add a few more comments on the previous result. First of all it is remarkable that Theorem 4.1 provides exact formulas for the various convex envelopes in the special setting of elastically rigid single-slip models. In general, explicit effective energy densities are very difficult to achieve with purely analytical methods, since one faces a nonlinear infinite-dimensional optimization problem. That is the reason why there are so few exact relaxations to be found in the literature.
Note the fundamental difference between the results in two and three space dimensions. While relaxation leads to an effective expansion of the kinematics in 2D, the three-dimensional situation is macroscopically quite rigid.
Moreover, the 3D results of Theorem 4.1 serve as an illustrative and physically relevant example that a quasiconvex extended-valued function need not necessarily be rank-one convex, or in other words that $W^{\mathrm{qc}} \leq W^{\mathrm{rc}}$ is not true. Indeed, this is essentially due to the postulate of rigid elasticity. As soon as the elastically rigid behavior is replaced by the assumption of an elastic energy with finite penalization of non-rotational strains, for example by substituting $\infty$ in the definition of $W$ by a large constant or by defining $W_{\mathrm{el}}$ as the squared distance from SO(3) multiplied with a large value, we no longer observe the previous rigidity phenomena. Instead, the corresponding relaxation will be again smaller than the rank-one convex envelope.

## 4.2. Introduction of a model with elastic energy

Although the elastically rigid regime introduced in Section 4.1 is convenient for the treatment with analytical tools, it is more natural from the physical viewpoint to have a model with finite elastic energy. For the rest of this thesis we will widely focus on the elastic energy density

$$W_{\mathrm{el},\varepsilon}(F_{\mathrm{el}}) = W_{\mathrm{el},\varepsilon;q}(F_{\mathrm{el}}) = \frac{1}{\varepsilon}\ \mathrm{dist}^q\left(F_{\mathrm{el}}(\mathbb{I} - \gamma s \otimes m), \mathrm{SO}(n)\right), \tag{4.16}$$

where $q \geq 1$, $\varepsilon > 0$ and $F_{\mathrm{el}} \in \mathbb{R}^{n\times n}$ with space dimension $n \in \{2,3\}$. It is immediate to see that the pointwise limit of $W_{\mathrm{el},\varepsilon}$ as $\varepsilon$ tends to zero yields exactly the rigid energy

density $W_{\mathrm{el}}^{\mathrm{rigid}}$ defined in (4.1), i.e. $\lim_{\varepsilon\to 0} W_{\mathrm{el},\varepsilon}(F) = W_{\mathrm{el}}^{\mathrm{rigid}}(F)$ for all $F \in \mathbb{R}^{n\times n}$. For comments on more general elastic energies we refer to Section 7.4.2.
Then, in view of (2.16) the condensed energy density of the first incremental problem reads

$$W_\varepsilon(F) = W_{\varepsilon;q,p}(F) = \inf_{\gamma\in\mathbb{R}} \left\{ \frac{1}{\varepsilon} \ \mathrm{dist}^q \left( F(\mathbb{I} - \gamma s \otimes m), \mathrm{SO}(n) \right) + |\gamma|^p \right\} \tag{4.17}$$

for all $F \in \mathbb{R}^{n\times n}$ with the elastic and plastic exponents $p, q \geq 1$. The parameter $\varepsilon$ is supposed to be small and can be interpreted as the ratio of the critical stress and the elastic constants.
The major goal of this work is to decide whether the rigid model of Section 4.1 can serve as a good approximation for the models with elastic energy provided $\varepsilon$ is small or speaking in terms of applications, if the elastic constants are large compared to the threshold of the resolved shear stress. Actually, the pointwise limit $W_\varepsilon \to W$ for $\varepsilon \to 0$, computed in Lemma 7.6, might suggest a positive answer. However, it turns out that this is not always true. In Chapter 7 we will see that the regularizing effect of hardening is crucial to approximate the models carrying an increasing amount of elastic energy for small elastic deformations by the assumption of rigid elasticity.

# 5. Preliminaries and technical tools

In this chapter we collect various technical tools and preliminaries which will be needed subsequently. Topics include auxiliary results for separately convex functions, the statement of the area and coarea formula, the fundamental solution of Laplace's equation and Green's function, the definition of Besov spaces, Bessel-potential spaces and Soblev-Slobodeckij spaces along with useful embedding and trace theorems, a few notions from real analysis, some key results on singular and fractional integrals as well as selected consequences of the convex integration method. We dispense with providing proofs in most cases, but refer to the quoted literature for the details and for further information on the subject.

## 5.1. Separately convex functions

Let us begin by giving the definition of separate convexity, which is another notion of semiconvexity, still weaker than rank-one convexity.

**Definition 5.1 (Separate convexity)** *A function $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}}$ is called separately convex, if*

$$f\left(\lambda F + (1-\lambda)G\right) \leq \lambda f(F) + (1-\lambda)f(G)$$

*for all $\lambda \in (0,1)$ and all $F, G \in \mathbb{R}^{m\times n}$ such that $F - G$ has only one non-zero entry.*

For the rest of this section we want to limit ourselves to real-valued functions. Then, any quasiconvex $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is also separately convex, for in case of real-valued maps quasiconvexity yields rank-one convexity, which in turn implies separate convexity.
We summarize here some important features of separately convex functions. The following statement is a quotation from [10, Lemma 2.2] and it says that the Lipschitz constant of a separately convex function is controlled locally by its oscillation.

**Lemma 5.2** *Let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be separately convex, $F \in \mathbb{R}^{m\times n}$ and $r > 0$. Then*

$$\operatorname{lip}(f; B(F,r)) \leq \sqrt{mn}\,\frac{\operatorname{osc}(f; B(F,2r))}{r},$$

*with $\operatorname{lip}(f; B(F,r))$ the Lipschitz constant of $f$ on $B(F,r)$ and the oscillation given by $\operatorname{osc}(f; B(F,r)) = \sup\{|f(X) - f(Y)| : X, Y \in B(F,r)\}$. In particular, a (real-valued) separately convex function is locally Lipschitz continuous.*

Next let us give a useful corollary [10, Corollary 2.3] of the previous lemma showing that differentiable and separately convex functions are in fact $C^1$.

**Corollary 5.3** *Let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be differentiable and separately convex. Then $\nabla f : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is continuous.*

Here is one more lemma taken from [10].

**Lemma 5.4** *Suppose $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is separately convex and $g : \mathbb{R}^{m\times n} \to \mathbb{R}$ is affine with $g(F) = f(F)$ for some $F \in \mathbb{R}^{m\times n}$. Then for any $r > 0$*

$$-\inf_{X\in B(F,r)} \big(f(X) - g(X)\big) \quad\leq\quad (2^{mn} - 1) \sup_{X\in B(F,r)} \big(f(X) - g(X)\big).$$

The property of functions called upper semidifferentiability is a weaker concept of differentiability and is introduced in the next definition. It will become clear by Corollary 5.7 that differentiability in the Frechet sense and lower semidifferentiability are equivalent for separately convex functions.

**Definition 5.5 (Upper semidifferentiability)** *A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be upper semidifferentiable in $F$, if there exists $A \in \mathbb{R}^{m\times n}$ such that*

$$\limsup_{X\to 0} \frac{f(F+X) - f(F) - A : X}{|X|} \leq 0. \tag{5.1}$$

*Further, $f$ is called upper semidifferentiable, if it is upper semidifferentiable in the foregoing sense for all $F \in \mathbb{R}^{m\times n}$.*

The subsequent result provides a large class of examples for upper semidifferentiable functions.

**Lemma 5.6** *Let $I$ be an index set and $f^i \in C^1(\mathbb{R}^{m\times n})$ for all $i \in I$. Further suppose that the infimum $\inf_{i\in I} f^i(F)$ is attained for all $F \in \mathbb{R}^{m\times n}$. Let the function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be defined by*

$$f(F) = \min_{i\in I} f^i(F).$$

*Then, $f$ is upper semidifferentiable on $\mathbb{R}^{m\times n}$.*

*PROOF.* Let $F \in \mathbb{R}^{m\times n}$. According to the definition of $f$ there is an element $i \in I$ such that $f(F) = f^i(F)$. So we get by the differentiability of $f^i$ that

$$\begin{aligned}\limsup_{X\to 0} &\frac{f(F+X) - f(F) - (\nabla f_i(F) : X)}{|X|} \\ &\leq \limsup_{X\to 0} \frac{f^i(F+X) - f^i(F) - \big(\nabla f^i(F) : X\big)}{|X|} = 0,\end{aligned}$$

which finishes the proof. □

As a consequence of Lemma 5.4 one gets the aforementioned relation that semidifferentiability and differentiability are equivalent notions regarding separately convex functions.

**Corollary 5.7** *Suppose $g : \mathbb{R}^{m\times n} \to \mathbb{R}$ is separately convex and assume $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is upper semidifferentiable at $F \in \mathbb{R}^{m\times n}$. Moreover let $g \leq f$ on $\mathbb{R}^{m\times n}$ and $g(F) = f(F)$. Then $f$ and $g$ are differentiable at $F$ and $\nabla f(F) = \nabla g(F)$.*

We conclude this collection of technical tools by the following elementary proposition for real convex functions, which will be needed later in Section 7.3.1.

**Proposition 5.8** *Let $[-\alpha, \alpha] \subset \mathbb{R}$, $f \in C^2([-\alpha, \alpha])$ and $\delta > 0$. Then for all convex functions $g \in C^1([-\alpha, \alpha])$ satisfying*

$$|f(x) - g(x)| < \delta \qquad \text{for all } x \in [-\alpha, \alpha] \tag{5.2}$$

*one has*

$$\left|f'(0) - g'(0)\right| < \frac{2\delta}{\alpha} + \alpha \max_{y\in[-\alpha,\alpha]} |f''(y)|.$$

*PROOF.* By the convexity of $g$ we get for all $x \in [-\alpha, \alpha]$ that

$$g(x) \geq g(0) + x\, g'(0).$$

Then requirement (5.2) provides $\delta + f(x) > f(0) - \delta + x\, g'(0)$. With the help of the mean value theorem this can be rewritten as

$$f'(\xi) > g'(0) - \frac{2\delta}{x} \tag{5.3}$$

for $x > 0$ with $\xi \in (0, x)$ and in the case $x < 0$ as

$$f'(\xi) < g'(0) - \frac{2\delta}{x} \tag{5.4}$$

with $\xi \in (x, 0)$. Again by the application of the mean value theorem it holds

$$f'(\xi) = f'(0) + \xi\, f''(\eta) \tag{5.5}$$

for some $\eta \in (0, \xi)$, if $\xi > 0$ and $\eta \in (\xi, 0)$, if $\xi < 0$. Plugging (5.5) into (5.3) and (5.4) yields

$$g'(0) - f'(0) \;<\; \frac{2\delta}{x} + x \max_{y\in[-\alpha,\alpha]} |f''(y)| \qquad \text{for } x > 0, \tag{5.6}$$

$$f'(0) - g'(0) \;<\; -\frac{2\delta}{x} - x \max_{y\in[-\alpha,\alpha]} |f''(y)| \qquad \text{for } x < 0. \tag{5.7}$$

The claim follows by specifying $x = \alpha$ in (5.6) and $x = -\alpha$ in (5.7). □

## 5.2. Area and Coarea formulas

The subsequent proposition gives the representation of summable functions in polar coordinates and is an immediate consequence of the well-known coarea formula, see [32, Section 3.4.2] or [5, Theorem 2.93].

**Proposition 5.9** *Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be $\mathcal{L}^n$-summable. Then*

$$\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x = \int_0^\infty \left( \int_{\partial B(0,r)} g(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) \right) \mathrm{d}r.$$

Next we state the area formula as it can be found in [5, Theorem 2.71].

**Theorem 5.10 (Area formula)** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function with $m \geq n$. Then, for any $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ the function $\mathcal{H}^0\big(E \cap f^{-1}\{.\}\big)$ is $\mathcal{H}^n$-measurable in $\mathbb{R}^m$ and*

$$\int_{\mathbb{R}^m} \mathcal{H}^0\big(E \cap f^{-1}\{y\}\big)\,\mathrm{d}\mathcal{H}^n(y) = \int_E J^{(n)} Df(x)\,\mathrm{d}x, \tag{5.8}$$

*where $f^{-1}\{y\}$ is the preimage of $y \in \mathbb{R}^m$ under $f$, $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$ is the differential of $f$ in $x$ and $J^{(n)}$ the $n$-dimensional Jacobian.*

Here the $n$-dimensional Jacobian is defined as follows.

**Definition 5.11** *Suppose $V$ and $W$ are Hilbert spaces with dim $V = n \leq$ dim $W$ and let $L : V \to W$ be a linear mapping. Then the $n$-dimensional Jacobian is defined as*

$$J^{(n)} L := \sqrt{\det\left(L^* \circ L\right)},$$

*where $L^* : W \to V$ is the adjoint of $L$.*

The following change of variables formula, which is a consequence of Theorem 5.10, will be needed later on.

**Proposition 5.12** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function with $m \geq n$. Then for any $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ and any $\mathcal{L}^n$-summable function $g : E \to [0, \infty)$ it holds*

$$\int_{\mathbb{R}^m} \left( \sum_{x \in E \cap f^{-1}\{y\}} g(x) \right) \mathrm{d}\mathcal{H}^n(y) = \int_E g(x) J^{(n)} Df(x)\,\mathrm{d}x. \tag{5.9}$$

*PROOF of Proposition 5.12.* The arguments used here follow the lines of the proof of [32, Section 3.3.3]. According to [32, Section 1.1.2, Theorem 7] there exist $\mathcal{L}^n$-measurable subsets $\{A^j\}_{j=1}^\infty$ of $E$ such that $g = \sum_{j=1}^\infty \frac{1}{j} \chi_{A^j}$, where $\chi_{A^j}$ denotes the characteristic function of $A^j$, i.e. $\chi_{A^j}(x) = 1$ for $x \in A^j$ and $\chi_{A^j}(x) = 0$ else. Using the monotone convergence theorem we infer from (5.8) that

$$\begin{aligned}
\int_E g(x) J^{(n)} Df(x)\, \mathrm{d}x &= \sum_{j=1}^{\infty} \frac{1}{j} \int_{A^j \cap E} J^{(n)} Df(x)\, \mathrm{d}x \\
&= \sum_{j=1}^{\infty} \frac{1}{j} \int_{\mathbb{R}^m} \mathcal{H}^0\big(E \cap A^j \cap f^{-1}\{y\}\big)\, \mathrm{d}\mathcal{H}^n(y) = \int_{\mathbb{R}^m} \sum_{j=1}^{\infty} \frac{1}{j}\, \mathcal{H}^0\big(E \cap A^j \cap f^{-1}\{y\}\big)\, \mathrm{d}\mathcal{H}^n(y) \\
&= \int_{\mathbb{R}^m} \sum_{j=1}^{\infty} \frac{1}{j} \Big( \sum_{x \in E \cap f^{-1}\{y\}} \chi_{A^j}(x) \Big)\, \mathrm{d}\mathcal{H}^n(y) = \int_{\mathbb{R}^m} \Big( \sum_{x \in E \cap f^{-1}\{y\}} g(x) \Big)\, \mathrm{d}\mathcal{H}^n(y).
\end{aligned}$$

This is exactly the desired formula. □

## 5.3. Function spaces of fractional order

This section is meant to provide the definitions of fractional Sobolev spaces, Besov spaces, Bessel-potential spaces and their relations as far as they are needed for the purpose of this thesis. We want to define thoroughly the corresponding boundary spaces and quote some embedding and trace theorems from [76], to which we also point for further results, proofs and details.
In this paragraph all functions are complex-valued unless stated otherwise. Besides, we denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ its dual, the space of tempered distributions on $\mathbb{R}^n$. The following definition is taken from [76, Section 2.3.1, Definition 1; Section 2.3.3].

**Definition 5.13 (The spaces $B^s_{p,q}$, $H^s_p$, $W^s_p$)**
*(i) For $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \le q < \infty$ the Besov space $B^s_{p,q}(\mathbb{R}^n)$ is defined by*

$$\begin{aligned}
B^s_{p,q}(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) \,\Big|\, f = \sum_{j=0}^{\infty} g^j \text{ with } \{g^j\}_{j \in \mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^n), \\
\operatorname{supp}(\mathcal{F} g^j) \subset M^j,\ \sum_{j=0}^{\infty} \big(2^{sj} \|g^j\|_{L^p(\mathbb{R}^n)}\big)^q < \infty \Big\},
\end{aligned} \tag{5.10}$$

*where $M^j = \overline{B(0, 2^{j+1})} \setminus B(0, 2^{j-1}) \subset \mathbb{R}^n$ for $j \in \mathbb{N}$ and $M^0 = \overline{B(0,2)} \subset \mathbb{R}^n$. Further, the space $B^s_{p,q}(\mathbb{R}^n)$ is endowed with the norm*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \inf \Big\{ \Big( \sum_{j=0}^{\infty} \big(2^{sj} \|g^j\|_{L^p(\mathbb{R}^n)}\big)^q \Big)^{1/q} \,\Big|\, f = \sum_{j=0}^{\infty} g^j \text{ with } g^j \text{ as in (5.10)} \Big\}.$$

*Notice that the Fourier transform* $\mathcal{F}g^j$ *of* $g^j \in \mathcal{S}'(\mathbb{R}^n)$ *with* $j \in \mathbb{N}_0$ *is compactly supported. So in view of the Paly-Wiener-Schwartz theorem, see [77, VI.4],* $g^j$ *is a regular distribution and can be represented by an analytic function, which is again denoted* $g^j$*.*
*(ii) For* $s \in \mathbb{R}$ *and* $1 < p < \infty$

$$H^s_p(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, \|\mathcal{F}^{-1}(1+|x|^2)^{s/2}\mathcal{F}f\|_{L^p(\mathbb{R}^n)} < \infty \right\}$$

*with the norm*

$$\|f\|_{H^s_p(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1+|x|^2)^{s/2}\mathcal{F}f\|_{L^p(\mathbb{R}^n)}$$

*are the Bessel-potential spaces.*
*(iii) If* $1 < p < \infty$ *and* $s \geq 0$ *we set*

$$W^s_p(\mathbb{R}^n) = \begin{cases} H^s_p(\mathbb{R}^n) & \text{for } s \in \mathbb{N}_0\,, \\ B^s_{p,p}(\mathbb{R}^n) & \text{for } s \in \mathbb{R}^+ \setminus \mathbb{N}. \end{cases}$$

*These spaces are named Sobolev-Slobodeckij spaces.*

**Remark 5.14** *1. It can be shown that the spaces* $W^s_p(\mathbb{R}^n)$*,* $H^s_p(\mathbb{R}^n)$ *and* $B^s_{p,q}(\mathbb{R}^n)$ *are Banach spaces [76, Section 2.3.2] and that they can be interpreted as subspaces of* $L^p(\mathbb{R}^n)$*. In particular, for* $k \in \mathbb{N}$ *the Sobolev-Slobodeckij spaces* $W^k_p(\mathbb{R}^n)$ *coincide with the well-known Sobolev spaces* $W^{k,p}(\mathbb{R}^n)$*, compare [76, Section 2.3.3].*
*2. The corresponding spaces for functions defined on an open set* $\Omega \subset \mathbb{R}^n$ *are obtained by restricting the whole-space versions in the sense of [76, Section 4.2.1, Definition 1]. Hence,* $B^s_{p,q}(\Omega)$ *is the restriction of* $B^s_{p,q}(\mathbb{R}^n)$ *to* $\Omega$ *with the norm reading*

$$\|f\|_{B^s_{p,q}(\Omega)} = \inf\left\{ \|g\|_{B^s_{p,q}(\mathbb{R}^n)} \,\middle|\, g \in B^s_{p,q}(\mathbb{R}^n),\ g|_\Omega = f \right\},$$

*where* $g|_\Omega \in \mathcal{D}'(\Omega)$ *is the restriction of* $g \in \mathcal{D}'(\mathbb{R}^n)$ *to* $\Omega$ *in the sense of distributions. The equality* $g|_\Omega = f$ *is to be understood in terms of distributions as well.*
*Analogously one defines* $W^s_p(\Omega)$ *and* $H^s_p(\Omega)$*.*

We decided to write down the following investigations regarding boundary spaces in terms of Besov spaces. However, it is clear that literally the same arguments apply, if $B^s_{p,q}(\mathbb{R}^n)$ is replaced by any of the other function spaces from Definition 5.13 or by some classical Sobolev space. Now we want to define function boundary spaces with the help of charts following the approach in [50, Chapter 1, Section 7.3] and [76, Section 3.6.1]. This will show why many of the properties valid for the function spaces on $\mathbb{R}^n$ carry over to those defined on the boundary.
Assume $\Omega \subset \mathbb{R}^n$ is an open and bounded set with smooth boundary, meaning that $\partial\Omega$ is $C^\infty$. For the introduction of Besov boundary spaces we start by choosing an open

covering $\{O^j\}_{j\in\{1,\ldots,N\}}$ of $\partial\Omega$ in such a way that for every $j \in \{1,\ldots,N\}$ there exists a bijective and infinitely differentiable mapping

$$\begin{array}{rcl} \varphi^j : O^j & \to & Q^n := (-1,1)^n \\ x & \mapsto & \varphi^j(x) \end{array}$$

with a smooth inverse

$$\begin{array}{rcl} (\varphi^j)^{-1} : Q^n & \to & O^j \\ y & \mapsto & (\varphi^j)^{-1}(y). \end{array}$$

In particular, it is required that $\varphi^j(O^j \cap \partial\Omega) = Q^n \cap \{y_n = 0\}$ and $\varphi^j(O^j \cap \Omega) = Q^n \cap \{y_n > 0\}$, where $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Moreover $\{\varphi^j\}_{j\in\{1,\ldots,N\}}$ are supposed to fulfill these compatibility conditions:
If $O^i \cap O^j \neq \emptyset$ for $i, j \in \{1,\ldots,N\}$, then there is a $C^\infty$-diffeomorphism

$$\begin{array}{rcl} \Psi^{ij} : \varphi^i(O^i \cap O^j) & \to & \varphi^j(O^i \cap O^j) \\ \varphi^i(x) & \mapsto & \Psi^{ij}(\varphi^i(x)) = \varphi^j(x) \end{array}$$

with $\det \nabla\Psi^{ij} > 0$.
Next let $\{\zeta^j\}_{j\in\{1,\ldots,N\}}$ be a partition of unity on $\partial\Omega$ with the properties $\zeta^j \in C^\infty(\partial\Omega;\mathbb{R})$, the space of infinitely differentiable functions on $\partial\Omega$, $\operatorname{supp}\zeta^j \subset O^j \cap \partial\Omega$ and $\sum_{j=1}^N \zeta^j = 1$ on $\partial\Omega$.
For $y' = (y_1, y_2, \ldots, y_{n-1}) \in Q^{n-1}$, $j \in \{1,\ldots,N\}$ and $u : \partial\Omega \to \mathbb{C}$ we define

$$\varphi^j_*(\zeta^j u)(y') = (\zeta^j u)\left((\varphi^j)^{-1}(y', 0)\right). \tag{5.11}$$

Since $\zeta^j$ has compact support in $O^j \cap \partial\Omega$, the function $\varphi^j_*(\zeta^j u)$ can be extended to $\mathbb{R}^{n-1}$ by 0. We introduce the space $B^s_{p,q}(\partial\Omega)$ by setting

$$B^s_{p,q}(\partial\Omega) = \{u : \partial\Omega \to \mathbb{C} \mid \varphi^j_*(\zeta^j u) \in B^s_{p,q}(\mathbb{R}^{n-1}) \text{ for } j = 1,\ldots,N\}. \tag{5.12}$$

The space $B^s_{p,q}(\partial\Omega)$ is endowed with the norm

$$\|u\|_{B^s_{p,q}(\partial\Omega)} = \sum_{j=1}^N \|\varphi^j_*(\zeta^j u)\|_{B^s_{p,q}(\mathbb{R}^{n-1})}, \tag{5.13}$$

which obviously depends on $\{(O^j, \varphi^j, \zeta^j)\}_{j\in\{1,\ldots,N\}}$. It can be shown, however, that all these norms are equivalent. For proofs and further details in the field of this topic we refer to [50]. In the same way we get $W^s_p(\partial\Omega)$, $H^s_p(\partial\Omega)$ with $s, p$ as in Definition 5.13 and the boundary Sobolev spaces $W^{k,p}(\partial\Omega)$ for $k \in \mathbb{N}$ and $1 \le p \le \infty$.

Here are some results for later use, which we quote from [76] in special versions adapted to our needs. First there are these two embedding results from [76, Section 4.6.1, Theorem (b)] and [76, Section 4.6.1, Remark 1].

**Theorem 5.15 (Embedding theorems)**
*(i) Let $\Omega \subset \mathbb{R}^n$ be open, $s \in \mathbb{R}$ and $2 \leq p < \infty$. Then the embedding*

$$H^s_p(\Omega) \hookrightarrow B^s_{p,p}(\Omega)$$

*is continuous.*
*(ii) If $0 \leq t \leq s < \infty$ and $1 < p \leq q < \infty$ with $s - \frac{n}{p} \geq t - \frac{n}{q}$, there is the continuous embedding*

$$W^s_p(\Omega) \hookrightarrow W^t_q(\Omega). \tag{5.14}$$

Next let us quote the following trace theorem from [76, Section 4.7.1, Theorem (b)].

**Theorem 5.16 (Trace theorem)** *Suppose $\Omega \subset \mathbb{R}^n$ is open, bounded and $\partial\Omega$ is $C^\infty$. Further let $1 < p < \infty$ and $1/p < s < \infty$. Then the trace operator*

$$\begin{array}{rcl} \gamma : B^s_{p,p}(\Omega) & \to & B^{s-\frac{1}{p}}_{p,p}(\partial\Omega) \\ f & \mapsto & \gamma(f) = f|_{\partial\Omega} \end{array}$$

*is continuous and features a continuous right inverse. Here $f|_{\partial\Omega}$ denotes the boundary value of $f$.*

**Remark 5.17** *1. Taking only those functions in $W^s_p(\mathbb{R}^n;\mathbb{C})$, $H^s_p(\mathbb{R}^n;\mathbb{C})$ and $B^s_{p,q}(\mathbb{R}^n;\mathbb{C})$ with real image one obtains the subspaces $W^s_p(\mathbb{R}^n;\mathbb{R})$, $H^s_p(\mathbb{R}^n;\mathbb{R})$ and $B^s_{p,q}(\mathbb{R}^n;\mathbb{R})$. The same can be done for the spaces defined on $\Omega \subset \mathbb{R}^n$ or for the function boundary spaces introduced above.*
*2. The foregoing definitions and results extend immediately to vector-valued (complex or real) functions considering single components.*

## 5.4. Fundamental solution of Laplace's equation and Green's function

For now we take $n \geq 2$ and denote the fundamental solution of Laplace's equation by $\Phi$. More precisely, for $x \in \mathbb{R}^n \setminus \{0\}$ we have

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x|, & \text{if } n = 2\,, \\ \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}, & \text{if } n \geq 3, \end{cases}$$

see for instance [31, Section 2.2.1] or [4, Section 8.17]. Then $\Phi \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$-\Delta\Phi = \delta_0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^n),$$

i.e. $\langle -\Delta\Phi, \varphi\rangle = \varphi(0)$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. So far for the problem on the whole space.
In the next few lines we want to motivate and state the formula of Green's function for an open and bounded set $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. For this purpose one introduces an appropriate corrector function to enforce the desired boundary conditions. For $x \in \Omega$ let $\phi_x$ denote the solution of the boundary value problem

$$\begin{aligned} -\Delta\phi_x &= 0 && \text{in } \Omega, \\ \phi_x(y) &= \Phi(y-x) && \text{for } y \in \partial\Omega. \end{aligned} \tag{5.15}$$

Since $\phi_x$ is harmonic and $\Phi$ is $C^\infty$-smooth away from 0, we have $\phi_x \in C^\infty(\overline{\Omega})$ for every $x \in \Omega$. Then, Green's function is defined for $x, y \in \Omega$ with $x \neq y$ through

$$G(x,y) = \Phi(y-x) - \phi_x(y).$$

Setting $G_x(y) = G(x,y)$ we obtain

$$\begin{aligned} -\Delta G_x &= \delta_x && \text{in } \mathcal{D}'(\Omega), \\ G_x &= 0 && \text{on } \partial\Omega \end{aligned}$$

for every $x \in \Omega$. It can be shown that Green's function is symmetric [31, Chapter 2.2], in formulas $G(x,y) = G(y,x)$. Hence, the equality $\Phi(x-y) = \Phi(y-x)$ for all $x, y \in \Omega$ with $x \neq y$ implies $\phi_x(y) = \phi_y(x)$ for $x, y \in \Omega$. With the definition

$$\phi(x,y) = \phi_x(y) = \phi_y(x) \tag{5.16}$$

and the associated smoothness of $\phi$ resulting from (5.15) we find for every $V \subset\subset \Omega$ a constant which bounds $\phi$ and all its derivatives uniformly on $V \times \Omega$.
In some of the subsequent chapters we will have to treat Dirichlet problems with a divergence term as a right-hand side. Our goal is now to determine a generalized Green's function for this case. This in mind let us first consider the situation on the whole space and take $\widetilde{\Phi} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ as

$$\widetilde{\Phi}(x) = \nabla\Phi(x) = -\frac{1}{n\omega_n}\frac{x}{|x|^n}.$$

Consequently, $\widetilde{\Phi} \in C^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ and

$$\langle \Delta\widetilde{\Phi}_j, \varphi\rangle = \langle -\Delta\Phi, \partial_j\varphi\rangle = \partial_j\varphi(0) = \big(\nabla\varphi(0)\big)_j \tag{5.17}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $j \in \{1, \dots, n\}$.
As before, for the investigation of the problem on a bounded set $\Omega$ a suitable corrector function is needed. This results from the next lemma, which provides an explicit representation formula for classical solutions of the boundary value problem we are interested in.

**Lemma 5.18** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\partial\Omega$ of class $C^\infty$ and let $f \in C^\infty(\overline{\Omega};\mathbb{R}^n)$. If $u \in C^2(\overline{\Omega})$ solves*

$$\begin{aligned} \Delta u &= \operatorname{div} f \qquad \text{in } \Omega, \\ u &= 0 \qquad\quad\ \ \text{on } \partial\Omega, \end{aligned}$$

*then for $x \in \Omega$,*

$$u(x) = \int_\Omega \widetilde{G}(x,y) \cdot f(y)\,\mathrm{d}y, \tag{5.18}$$

*where $\widetilde{G}(x,y) = \widetilde{\Phi}(y-x) - \nabla\phi_x(y)$ for $x, y \in \Omega$ with $x \neq y$.*

*PROOF.* From the classical representation formula by Green's function [31, Section 2.2.4, Theorem 12] for Poisson's equation with prescribed boundary data one deduces using integration by parts that

$$\begin{aligned} u(x) &= -\int_\Omega G(x,y)\operatorname{div} f(y)\,\mathrm{d}y = \int_\Omega \nabla G_x(y)\cdot f(y)\,\mathrm{d}y \\ &= \int_\Omega \big(\widetilde{\Phi}(y-x) - \nabla\phi_x(y)\big)\cdot f(y)\,\mathrm{d}y, \end{aligned}$$

which is (5.18). □

## 5.5. A few notions from real analysis

Next the notion of weak-$L^p$ spaces is introduced. These spaces are of great importance for our discussion of $L^p$-theory when focusing on $p = 1$ in Section 6.3. There we will see that the case $p = 1$ is essentially different from $p > 1$, since one can only expect weak-$L^1$ estimates instead of control in $L^1$.

Let us establish first the distribution function $\lambda_f$ of a measurable function $f : \Omega \to \mathbb{R}^m$, where $\Omega \subset \mathbb{R}^n$ is open. That is,

$$\begin{aligned} \lambda_f : [0,\infty) &\to \overline{\mathbb{R}} \\ t &\mapsto \lambda_f(t) = \big|\big\{x \in \Omega : |f(x)| > t\big\}\big|. \end{aligned} \tag{5.19}$$

This non-increasing and right-continuous function constitutes a good quantitative measure for the growth behavior of $f$ and helps to give an elegant definition of weak Lebesgue spaces. Here we follow [39, Definition 6.3] and extend the definition to vector-valued functions.

**Definition 5.19 (Weak-$L^p$ spaces)** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p < \infty$. A measurable function $f : \Omega \to \mathbb{R}^m$ is said to be in $L^p_w(\Omega;\mathbb{R}^m)$, if*

$$\sup_{t>0}\ t^p \lambda_f(t) < \infty.$$

*In particular, one defines* $\|f\|_{L^p_w(\Omega;\mathbb{R}^m)} = \sup_{t>0}\ t\big(\lambda_f(t)\big)^{1/p}$.

**Remark 5.20** *1. By definition,* $L^p_w(\Omega;\mathbb{R}^m)$*-functions are exactly those whose distribution functions decay at least with order* $p$.
*2. Note that* $\|\,.\,\|_{L^p_w(\Omega;\mathbb{R}^m)}$ *is not a norm on* $L^p_w(\Omega;\mathbb{R}^m)$*, since the triangle inequality is only correct up to an additional constant. Precisely, for* $f,g \in L^p_w(\Omega;\mathbb{R}^m)$ *one obtains*

$$\|f+g\|_{L^p_w(\Omega;\mathbb{R}^m)} \leq 2\,\Big(\|f\|_{L^p_w(\Omega;\mathbb{R}^m)} + \|g\|_{L^p_w(\Omega;\mathbb{R}^m)}\Big)\,.$$

*3. It holds* $L^p(\Omega;\mathbb{R}^m) \subsetneq L^p_w(\Omega;\mathbb{R}^m)$ *for* $p \geq 1$*. If* $\Omega \subset \mathbb{R}^n$ *is additionally bounded, one can show* $L^p(\Omega;\mathbb{R}^m) \subset L^p_w(\Omega;\mathbb{R}^m) \subset L^q(\Omega;\mathbb{R}^m)$ *for* $1 \leq p,q < \infty$ *with* $q < p$.
*4. For vector-valued functions* $f : \Omega \to \mathbb{R}^m$ *the spaces* $L^p_w(\Omega;\mathbb{R}^m)$ *can also be defined componentwise using Definition 5.19 for the real-valued components* $f_1,\ldots,f_m$*. In view of the estimate*

$$\frac{1}{m}\sup_{t>0}\ t\big(\lambda_f(t)\big)^{1/p} \leq \sum_{j=1}^{m}\ \sup_{t>0} t\big(\lambda_{f_j}(t)\big)^{1/p} \leq m\sup_{t>0}\ t\big(\lambda_f(t)\big)^{1/p}$$

*we see that these two ways of defining* $L^p_w(\Omega;\mathbb{R}^m)$ *are in fact equivalent.*
*5. A sequence* $\{f^k\}_{k\in\mathbb{N}} \subset L^p_w(\Omega;\mathbb{R}^m)$ *is said to converge to zero in* $L^p_w$ *for* $k \to \infty$*, in symbols*

$$f^k \to 0 \quad in \ \ L^p_w(\Omega;\mathbb{R}^m) \ \ as\ k \to \infty,$$

*if* $\lim_{k\to\infty}\|f^k\|_{L^p_w(\Omega;\mathbb{R}^m)} = 0$ *or equivalently, if* $\{f^k\}_{k\in\mathbb{N}}$ *satisfies the requirement that for every* $\delta > 0$ *there exists* $N = N(\delta)$ *such that*

$$t\ \big|\{x \in \Omega : |f^k(x)| > t\}\big|^{1/p} \leq \delta$$

*for all* $t > 0$ *and* $k \geq N$.

Finally we introduce the notion of strong and weak type $(p,q)$ for functions mapping $L^p(\mathbb{R}^n;\mathbb{R}^m)$ to $L^q(\mathbb{R}^n;\mathbb{R}^m)$ with $1 \leq p,q < \infty$. The generalization to vector-valued maps being obvious we formulate the following definition for real-valued functions only.

**Definition 5.21 (Strong and weak type (p,q))** *Let* $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ *with* $1 \leq p,q < \infty$*. Then* $T$ *is of (strong) type* $(p,q)$*, if*

$$\|T(f)\|_{L^q(\mathbb{R}^n)} \leq c\,\|f\|_{L^p(\mathbb{R}^n)} \tag{5.20}$$

*for all* $f \in L^p(\mathbb{R}^n)$*, where* $c > 0$ *is a constant independent of* $f$*. Further,* $T$ *is a map of weak type* $(p,q)$*, if*

$$\big|\,\{x \in \mathbb{R}^n : |T(f)(x)| > t\}\,\big| \leq \left(\frac{c\,\|f\|_{L^p(\mathbb{R}^n)}}{t}\right)^q \tag{5.21}$$

*for all* $f \in L^p(\mathbb{R}^n)$ *and* $t > 0$ *with* $c > 0$ *not depending on* $f$ *and* $t$.

Notice that in view of Definition 5.19 the estimate in (5.21) is equivalent to

$$\|T(f)\|_{L^q_w(\mathbb{R}^n)} \leq c\,\|f\|_{L^p(\mathbb{R}^n)}$$

for all $f \in L^p(\mathbb{R}^n)$.

## 5.6. Singular and fractional integrals

Taking [72, 39] as references we want to summarize some of the most important definitions and results in the context of convolution operators involving singular kernels whose only singularities are at the origin and at infinity. One special class of singular kernels with good properties are the Caldéron-Zygmund kernels. These are integrable in the Cauchy principle sense with respect to all neighborhoods of 0.

**Definition 5.22 (Calderón-Zygmund kernel)** *A function $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is said to be a Calderón-Zygmund kernel, if*

*(i) $k$ is homogenous of degree $-n$, i.e. for all $x \in \mathbb{R}^n \setminus \{0\}$ $k$ takes the form $k(x) = \frac{\omega(x)}{|x|^n}$, where $\omega$ is homogenous of degree $0$,*

*(ii) $k|_{\partial B(0,1)} = \omega$ and $\omega \in L^\infty(\mathbb{R}^n)$,*

*(iii) $\omega$ satisfies the cancelation property, that is $\displaystyle\int_{\partial B(0,1)} \omega \; d\mathcal{H}^{n-1} = 0.$*

For $\varepsilon > 0$ and $f \in L^p(\mathbb{R}^n)$ with $p \geq 1$ let us define $T_\varepsilon(f)(x) = \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} k(x-y)f(y)\,\mathrm{d}y$ for $x \in \mathbb{R}^n$. Further we set

$$\int_{\mathbb{R}^n} k(x-y)f(y)\,\mathrm{d}y := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} k(x-y)f(y)\,\mathrm{d}y = \lim_{\varepsilon \to 0} T_\varepsilon(f)(x) \tag{5.22}$$

provided the limit exists. An expression like (5.22) is what one usually understands by a singular integral.
The fundamental result in this section is the celebrated theorem by Calderón-Zygmund, see [39, Theorem 7.15] and [72, Section 4.2], which says that singular integrals regarded as mappings between $L^p$-spaces are of strong type $(p,p)$, if $1 < p < \infty$. For the case $p = 1$, however, one can only obtain weak type $(1,1)$ estimates for $T_\varepsilon$.

**Theorem 5.23 (Calderón-Zygmund theorem)** *Let $k$ be a Calderón-Zygmund kernel and suppose that its restriction to $\partial B(0,1)$ is Lipschitz continuous.*

*(i) Then $T_\varepsilon$ is of weak type $(1,1)$ uniformly in $\varepsilon$, i.e. there is a constant $c > 0$ not depending on $\varepsilon$ and $f$ such that*

$$\|T_\varepsilon(f)\|_{L^1_w(\mathbb{R}^n)} \leq c\,\|f\|_{L^1(\mathbb{R}^n)}$$

*for all $f \in L^1(\mathbb{R}^n)$.*

*(ii)* *Let $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$ and $\varepsilon > 0$. Then $T_\varepsilon(f) \in L^p(\mathbb{R}^n)$ together with the estimate*

$$\|T_\varepsilon(f)\|_{L^p(\mathbb{R}^n)} \leq c\, \|f\|_{L^p(\mathbb{R}^n)},$$

*where $c = c(p)$ is independent of $f$ and $\varepsilon$.*

*(iii)* *If $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$, the limit $T(f) := \lim_{\varepsilon \to 0} T_\varepsilon(f)$ exists in the sense of $L^p(\mathbb{R}^n)$ and is of strong type $(p, p)$.*

Actually, statements similar to those in Theorem 5.23 hold true for the maximal singular integral

$$T_* f(x) := \sup_{\varepsilon > 0} |T_\varepsilon f(x)|, \tag{5.23}$$

which can be used to prove pointwise convergence of $T_\varepsilon f$ almost everywhere in $\mathbb{R}^n$. For the exact formulation of these results see the following theorem or [39, Theorem 7.16], [72, Sections 4.5, 4.6].

**Theorem 5.24** *Under the assumptions of Theorem 5.23 the map $T_* : f \mapsto T_*(f)$ as defined in* (5.23) *is of weak type* $(1, 1)$ *and of strong type* $(p, p)$ *for* $1 < p < \infty$. *Moreover, for $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$ the limit*

$$\lim_{\varepsilon \to 0} T_\varepsilon(f)(x) \tag{5.24}$$

*exists for almost all $x \in \mathbb{R}^n$.*

The pointwise limit in (5.24), which coincides with the $L^p$-limit of Theorem 5.23 (iii) almost everywhere in $\mathbb{R}^n$ for $p > 1$, is usually denoted $T(f)$. Besides, this limit is exactly what we mean by $\int_{\mathbb{R}^n} k(x - y) f(y)\, \mathrm{d}y$. Therefore we will be using equivalently,

$$T(f)(x) = \int_{\mathbb{R}^n} k(x - y) f(y)\, \mathrm{d}y = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} k(x - y) f(y)\, \mathrm{d}y = \lim_{\varepsilon \to 0} T_\varepsilon(f)(x) \tag{5.25}$$

for almost all $x \in \Omega$ and $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$.

Finally let us say a word on fractional integrals, or more precisely on Riesz potentials.

**Definition 5.25 (Riesz-potential)** *For $f \in L^p(\mathbb{R}^n)$ and $0 < \alpha < n$ with $1 \leq p < \frac{n}{\alpha}$ the function $I_\alpha(f) : \mathbb{R}^n \to \mathbb{R}$ given by*

$$I_\alpha(f)(x) = \frac{\Gamma(\frac{n}{2} - \frac{\alpha}{2})}{\pi^{n/2} 2^\alpha \Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}}\, \mathrm{d}y,$$

*with $\Gamma : \mathbb{R}_0^+ \to \mathbb{R}$ the Euler function, is well defined almost everywhere in $\mathbb{R}^n$ and called the Riesz potential of $f$ of order $\alpha$.*

The next theorem contains useful statements on the boundedness of maps $f \mapsto I_\alpha(f)$ from $L^p$ into $L^q$ depending of the relation between $p$, $q$ and $\alpha$.

**Theorem 5.26 (Hardy-Littlewood-Sobolev inequality)** *Let $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$ and $q = \frac{np}{n-\alpha p}$. If $p = 1$, there is a constant $c = c(n,\alpha) > 0$ such that*

$$\|I_\alpha(f)\|_{L^q_w(\mathbb{R}^n)} \le c\,\|f\|_{L^1(\mathbb{R}^n)}$$

*for all $f \in L^1(\mathbb{R}^n)$. If $p > 1$, the map $f \mapsto I_\alpha(f)$ is of strong type $(p,q)$, i.e. there is $c = c(n,\alpha,p) > 0$ with*

$$\|I_\alpha(f)\|_{L^q(\mathbb{R}^n)} \le c\,\|f\|_{L^p(\mathbb{R}^n)}$$

*for all $f \in L^p(\mathbb{R}^n)$.*

## 5.7. Consequences of the convex integration method

Convex integration, which was originally introduced by Gromov [41] and later worked out in the Lipschitz setting by Müller and Šverák [64, 63] is a powerful tool for solving partial differential relations. That means to find a Lipschitz map $u : \Omega \to \mathbb{R}^m$ satisfying $\nabla u \in K \subset \mathbb{R}^{m\times n}$ almost everywhere in $\Omega \subset \mathbb{R}^n$ subject to a boundary constraint $u = v$ on $\partial\Omega$, where $v \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ is given.
Here we want to state only those results connected with convex integration we are going to use in the following. Let us start by quoting [64, Lemma 2.1], where $K$ is a neighborhood of two matrices whose difference has rank one.

**Lemma 5.27** *Let $\Omega \subset \mathbb{R}^n$ be bounded and $A_1, A_2 \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A_2 - A_1) = 1$. Further suppose that $C = \lambda A_1 + (1-\lambda)A_2$ for some $\lambda \in (0,1)$. Then for any $\delta \in (0, \frac{|A_2 - A_1|}{2})$ there exists a piecewise linear function $u : \Omega \to \mathbb{R}^m$ such that*

*(i)* $\nabla u \in B(A_1,\delta) \cup B(A_2,\delta)$ *almost everywhere in $\Omega$,*

*(ii)* $\sup_{x\in\Omega} |u(x) - Cx| \le \delta$,

*(iii)* $u(x) = Cx$ *on $\partial\Omega$,*

*(iv)* $\big|\{x \in \Omega \,|\, \nabla u(x) \in B(A_1,\delta)\}\big| = \lambda|\Omega|$.

**Remark 5.28** *Assertion (iv) does not explicitly occur in the formulation of [64, Lemma 2.1]. However, the proof given there is based on a construction done in such a way that the set where $\nabla u$ is close to $A_1$ has actually measure $\lambda|\Omega|$ provided $\delta$ is sufficiently small. In order to highlight this specific feature we repeat here part of the proof.*

*PROOF.* The idea of the proof is to construct a solution on a polyhedral building block $U$ first and then to apply Vitali's covering theorem to extend it to $\Omega$ by rescaling and translating copies of $U$.
After appropriate change of variables it is sufficient to restrict our investigations to

$$A_1 = (1-\lambda)a \otimes e_n, \qquad A_2 = -\lambda a \otimes e_n \quad \text{and} \quad C = 0,$$

where $e_n$ is the $n$-th standard basis vector in $\mathbb{R}^n$ and $a \in \mathbb{R}^m$ with $|a| = 1$.
Take $\delta \in \big(0, \frac{|A_2-A_1|}{2}\big)$ and $\varepsilon > 0$ (to be chosen later). Let $V$ be the cuboid $V = (-1,1)^{n-1} \times \big((\lambda-1)\varepsilon, \lambda\varepsilon\big)$. Further we define $v : V \to \mathbb{R}^m$ to be the affine function with $\nabla v = A_2$ in $V \cap \{x_n < 0\}$, $\nabla v = A_1$ in $V \cap \{x_n > 0\}$ and $v = 0$ for $x_n = (\lambda-1)\varepsilon$ and $x_n = \lambda\varepsilon$. In formulas this is

$$v(x) = -\varepsilon\lambda(1-\lambda)a + \begin{cases} (1-\lambda)x_n a, & \text{if } x_n \geq 0, \\ -\lambda x_n a, & \text{if } x_n < 0. \end{cases}$$

Just to remark, by construction and the choice of $\delta$ we have

$$\begin{aligned} \big|\{x \in V \,|\, \nabla v(x) \in B(A_1,\delta)\}\big| = \big|\{x \in V \,|\, \nabla v(x) = A_1\}\big| &= \lambda|V|, && (5.26) \\ \big|\{x \in V \,|\, \nabla v(x) \in B(A_2,\delta)\}\big| = \big|\{x \in V \,|\, \nabla v(x) = A_2\}\big| &= (1-\lambda)|V|. && (5.27) \end{aligned}$$

In the next step $v$ has to be modified to obtain a function with zero boundary values. To this end we set $w = v + h$ with $h(x) = \varepsilon\lambda(1-\lambda)(\sum_{i=1}^{n-1}|x_i|)a$ for $x \in V$. Notice that $w$ is piecewise affine and $w \cdot a \geq 0$ on $\partial V$, while $w(0) \cdot a < 0$. If $\varepsilon$ is small enough, it follows

$$\operatorname{dist}(\nabla w, \{A_1, A_2\}) \leq |\nabla h| = \varepsilon\lambda(1-\lambda)\sqrt{n-1} \leq \delta \tag{5.28}$$

almost everywhere in $V$ and

$$|w| \leq \varepsilon\lambda(1-\lambda)(n+1) \leq \delta \tag{5.29}$$

in $V$. The ratio of the points with $\nabla v$ close to $A_1$ and those where $\nabla v$ is close to $A_2$ from (5.26) and (5.27) is preserved when passing to $w$, since the gradients of the perturbed function $w$ are concentrated in disjoint $\delta$-balls around $A_1$ and $A_2$ in view of (5.28).
By defining $U = \{x \in V \,|\, w(x) \cdot a < 0\}$ and restricting $w$ to $U$ one eventually finds a piecewise linear function which fulfills the required assertions (i)-(iii) for $\Omega = U$. Indeed, (i) and (ii) result immediately from (5.28) and (5.29), respectively, while (iii) can be concluded recalling that $w$ and $a$ are parallel, so that $w = 0$ on $\partial U$. Geometrically $U$ is an open polyhedron in $\mathbb{R}^n$ whose corners are exactly the points where $V$ cuts the coordinate axes and it holds $|U| = 1/2|V|$. By construction we find $\big|\{x \in U | \nabla w(x) \in B(A_1,\delta)\}\big| = \lambda|U|$ and hence (iv) for $w$ on $U$.
Finally, we have to cover $\Omega$ with scaled copies of $U$ to achieve the claim. For the detailed argumentation of this last step we refer to the proof of [64, Lemma 2.1]. $\square$

This section provides the fundamental tools needed for the construction of recovery sequences for the limsup-inequality later in Theorem 7.18. The proof of this result essentially uses the method of convex integration. Here we want to state [64, Theorem 1.3] in a reformulated version as it can be found in [24, Theorem 3].

**Theorem 5.29** *Let $\Sigma = \{F \in \mathbb{R}^{n\times n} | \det F = 1\}$ and $K \subset \Sigma$. Suppose that $\{U^j\}_{j\in\mathbb{N}}$ is an in-approximation of $K$, i.e. the $U^j$ are open in $\Sigma$, uniformly bounded, $U^j$ is contained in the rank-one convex hull of $U^{j+1}$ and $U^j$ converges to $K$ in the sense that $F^j \to F$ as $j \to \infty$ with $F^j \in U^j$ implies $F \in K$.*
*Under these conditions there exists for any $F \in U^1$ and any bounded domain $\Omega \subset \mathbb{R}^n$ a Lipschitz solution of the partial differential inclusion*

$$\begin{aligned} Du &\in K \quad \text{almost everywhere in } \Omega, \\ u(x) &= Fx \quad \text{on } \partial\Omega. \end{aligned}$$

Both of the remaining results are merely two-dimensional and closely related to the single-slip models of crystal plasticity introduced in Chapter 4. Recall from (4.4) that $\mathcal{M}^{(2)}$ denotes the set where the 2D elastically rigid energy densities are finite, i.e. $\mathcal{M}^{(2)} = \{F \in \mathbb{R}^{2\times 2} \mid \det F = 1, \ |Fs| = 1\}$. Further $\mathcal{N}^{(2)}$ as defined in (4.6) is exactly its rank-one convex hull.
Let us point out a lemma which is taken from [24] and actually a corollary of Theorem 5.29. For the readers' convenience we give a detailed proof.

**Lemma 5.30** *Suppose $\Omega \subset \mathbb{R}^2$ is open and bounded, $k > 0$ and $U^k = \{G \in \mathcal{N}^{(2)} : |Gm|^2 < k^2\}$. Then for each $F \in U^k$ there exists a $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ such that $u = Fx$ on $\partial\Omega$ and $\nabla u \in U^k \cap \mathcal{M}^{(2)}$ almost everywhere in $\Omega$.*

*PROOF.* If $F \in U^k \cap \mathcal{M}^{(2)}$, the assertion follows by taking $u(x) = Fx$ for all $x \in \Omega$. Otherwise, namely for $F \in U^k \setminus \mathcal{M}^{(2)}$, Theorem 5.29 has to be applied. For this purpose we define for $j \in \mathbb{N}$,

$$U^{k,j} = \{G \in \mathbb{R}^{2\times 2} \mid \det G = 1, \ 1 - \frac{1}{2^{j-1}} < |Gs| < 1, \ |Gm|^2 < k^2\}.$$

First we will show that $\{U^{k,j}\}_{j\in\mathbb{N}}$ is an in-approximation for $K^k \cap \mathcal{M}^{(2)}$, where $K^k = \{G \in \mathcal{N}^{(2)} : \ |Gm|^2 \leq k^2\}$. Obviously, $U^{k,j}$ are open in $\Sigma = \{G \in \mathbb{R}^{2\times 2} \mid \det G = 1\}$ and uniformly bounded by $k+1$. Moreover $U^{k,j}$ converge to $K^k \cap \mathcal{M}^{(2)}$ in the desired way, i.e. if $G^j \in U^{k,j}$ and $G^j \to G$, then the properties of $G^j$ devolve to $G$ so that finally $G \in K^k \cap \mathcal{M}^{(2)}$. To verify that $U^{k,j} \subset (U^{k,j+1})^{\mathrm{rc}}$ let $G \in U^{k,j} \setminus U^{k,j+1}$. For $t \in \mathbb{R}$ we define

$$G_t = G + t(Gm \otimes s),$$

so that $\det G_t = \det G = 1$ , $|G_t m| = |Gm|$ and $|G_t s|^2 = |Gs|^2 + 2t\, Gs \cdot Gm + t^2 |Gm|^2$. By assumption it holds $|Gs| \in \big(1 - 2^{-j+1}, 1 - 2^{-j}\big]$. With $\alpha \in (1 - 2^{-j}, 1)$ we obtain $|Gs|^2 - \alpha < 0$ and the equation

$$|Gm|^2 t^2 + 2t\, Gs \cdot Gm + |Gs|^2 - \alpha = 0$$

has exactly two solutions with different signs. These are $t^- < 0 < t^+$. Consequently, $G_{t^+}, G_{t^-} \in U^{k,j+1}$, $\operatorname{rank}(G_{t^+} - G_{t^-}) = 1$ and there is a $\lambda \in (0,1)$ such that $G = \lambda G_{t^+} + (1-\lambda) G_{t^-}$. So $G$ can be obtained as a simple laminate supported in $U^{k,j+1}$ or in other words, $G$ can be written as a convex combination between two rank-one connected matrices in $U^{k,j+1}$.
Hence, $\{U^{k,j}\}_{j\in\mathbb{N}}$ is in fact the postulated in-approximation for $K^k \cap \mathcal{M}^{(2)}$, so that Theorem 5.29 yields for every $F \in U^{k,1} = U^k \setminus \mathcal{M}^{(2)}$ and every bounded domain $\Omega \subset \mathbb{R}^2$ the existence of $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ with $u(x) = Fx$ on $\partial\Omega$ and $\nabla u \in K^k \cap \mathcal{M}^{(2)}$.
In order to finish the proof it suffices to realize that for every $F \in U^{k,1}$ there is a small $\delta > 0$ such that $F \in U^{k-\delta,1}$. With the same argumentation as above applied for $k - \delta$ instead of $k$, one finally obtains a Lipschitz continuous function $u$ with $u = Fx$ on $\partial\Omega$ and $\nabla u \in K^{k-\delta} \cap \mathcal{M}^{(2)} \subset U^k \cap \mathcal{M}^{(2)}$, which is the assertion. □

It is necessary to stress once again that Lemma 5.30 only holds in the 2D setting. If it was true in the three-dimensional situation as well, this would yield a contradiction to [24, Theorem 2].
The rest of this section is concerned with simple laminate constructions in 2D. By a simple laminate of period $h > 0$ between two rank-one matrices $A, B \in \mathbb{R}^{2\times 2}$ with $A - B = a \otimes \nu$, where $a, \nu \in \mathbb{R}^2$, we mean a function $l : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$,

$$l(x) = \big(\lambda A + (1-\lambda) B\big)x + h\chi_\lambda\Big(\frac{\nu \cdot x + c}{h}\Big)\, a.$$

Here $\lambda$ and $(1-\lambda)$ are the weights of the laminate, $c \in \mathbb{R}$ describes shift and $\chi_\lambda$ is a continuous, one-periodic real-valued function of one variable with mean value zero on $(0,1)$ such that $\chi'_\lambda(t) = 1 - \lambda$ for $t \in (0,\lambda)$ and $\chi'_\lambda(t) = -\lambda$ for $t \in (\lambda, 1)$.
One of the main difficulties in working with simple laminates is to make them compatible with boundary conditions. In [64, Theorem 6.1] Müller and Šverák contributed to this line of thought by providing approximating piecewise linear maps with affine boundary data. Conti and Theil [24, Theorem 4] gave an even stronger result in that they were able to keep the exact laminate in a large part of the set and to enforce an additional convex constraint. The proof of the latter relies on an explicit piecewise construction, which can be found in the appendix of [24].

**Theorem 5.31** *Let $A, B \in \mathbb{R}^{2\times 2}$ with* $\det A = \det B = 1$ *and* $\operatorname{rank}(A - B) = 1$. *Further let $v \in \mathbb{R}^2$ be such that $|Av| = |Bv|$ and $Av \neq Bv$ and suppose $\Omega \subset \mathbb{R}^2$ is open. For any $\lambda \in (0,1)$ and any $\delta > 0$ there are $h_0 > 0$ and $\Omega_\delta \subset \Omega$ with $|\Omega \setminus \Omega_\delta| \le \delta$ such that the* *restriction to $\Omega_\delta$ of any simple laminate between the gradients $A$ and $B$ with weights $\lambda$ and $1-\lambda$ and period $h < h_0$ can be extended to a finitely piecewise affine function $u : \Omega \to \mathbb{R}^2$ so that $u(x) = \big(\lambda A + (1-\lambda)B\big)x$ for $x \in \partial\Omega$, $\det \nabla u = 1$, $|(\nabla u)v| \leq |Av| = |Bv|$ and $\operatorname{dist}(\nabla u, [A, B]) \leq \delta$ on $\Omega$.*

The property of $u : \Omega \to \mathbb{R}^2$ being finitely piecewise affine means that $\Omega$ can be decomposed into finitely many pieces such that $u$ is affine on each of them.
Referring back to the sets $\mathcal{M}^{(2)}$ and $\mathcal{N}^{(2)}$ Theorem 5.31 proves the assertion that for any simple laminate between rank-one matrices $A, B \in \mathcal{M}^{(2)}$ there exist extensions with affine boundary conditions in form of a piecewise affine function with gradients in $\mathcal{N}^{(2)}$.
By combining Theorem 5.31 and Lemma 5.30 we can show even more. A simple laminate between rank-one matrices $A, B \in \mathcal{M}^{(2)}$ can be extended to a Lipschitz function $u : \Omega \to \mathbb{R}^2$ with affine boundary data whose gradient $\nabla u$ is in $\mathcal{M}^{(2)}$ almost everywhere in $\Omega$.

# 6. Collection of auxiliary results

The intention of the following paragraphs is to collect the auxiliary results that will be needed later in Chapter 7 and 8 to analyze the models introduced in Section 4.2. In this spirit we begin by presenting several lemmata from the field of function spaces. This is followed by sections dealing with aspects of $L^p$-theory for special Dirichlet problems and by lemmata on properties of equi-integrable functions. Finally, in Section 6.5 we formulate and prove a theorem on compensated compactness which is the key tool for the verification of the $\Gamma$-convergence result pointed out in Theorem 1.2. To be more precise, Theorem 6.20 is a generalization of the classical div-curl lemma to sequences whose divergence and curl are compact as functionals on Lipschitz functions. It turns out to be the crucial ingredient in the proof of Theorem 7.18 for the recovery of the incompressibility constraint in the limit, but due to its generality it is also of interest itself.

## 6.1. Two lemmata for Sobolev boundary spaces

The subsequent lemma is a consequence of the representation of functions via polar coordinates, see Proposition 5.9.

**Lemma 6.1** *Let $p \geq 1$ and let $B(0,1)$ be the unit ball in $\mathbb{R}^n$. Then for every $u \in W^{1,p}(B(0,1))$ there exists a $\rho \in (1/2, 3/4)$ such that $u \in W^{1,p}(\partial B(0,\rho))$ and*

$$\|u\|_{W^{1,p}(\partial B(0,\rho))} \leq c\,\|u\|_{W^{1,p}(B(0,1))}$$

*with a constant $c > 0$ depending only on $p$ and $n$.*

*PROOF.* We assert that for every $u \in W^{1,p}(B(0,1))$ there is a set $M \subset (1/2, 3/4)$ with $|M| > 0$ such that

$$\int_{\partial B(0,\rho)} |u(x)|^p + |\nabla u(x)|^p \,\mathrm{d}\mathcal{H}^{n-1}(x) \leq 4\Big(\|u\|^p_{L^p(B(0,1))} + \|\nabla u\|^p_{L^p(B(0,1);\mathbb{R}^n)}\Big) \tag{6.1}$$

for all $\rho \in M$. Suppose the claim was wrong, then

$$\int_{1/2}^{3/4}\Big(\int_{\partial B(0,\rho)}|u(x)|^p + |\nabla u(x)|^p \,\mathrm{d}\mathcal{H}^{n-1}(x)\Big)\,\mathrm{d}\rho > \|u\|^p_{L^p(B(0,1))} + \|\nabla u\|^p_{L^p(B(0,1);\mathbb{R}^n)}. \tag{6.2}$$

On the other hand, by using Proposition 5.9 with $g(x) = \chi_{B(0,3/4)\setminus B(0,1/2)}(x)\big(|u(x)|^p + |\nabla u(x)|^p\big)$ for $x \in \mathbb{R}^n$ we find

$$\begin{aligned}\int_{1/2}^{3/4} \Big( \int_{\partial B(0,\rho)} &|u(x)|^p + |\nabla u(x)|^p \,\mathrm{d}\mathcal{H}^{n-1}(x) \Big)\,\mathrm{d}\rho \\ &= \int_{B(0,3/4)\setminus B(0,1/2)} |u(x)|^p + |\nabla u(x)|^p \,\mathrm{d}x \le \|u\|^p_{L^p(B(0,1))} + \|\nabla u\|^p_{L^p(B(0,1);\mathbb{R}^n)}.\end{aligned}$$

This is in contradiction to (6.2).
It remains to show that $\int_{\partial B(0,\rho)} |u|^p + |\nabla u|^p \,\mathrm{d}\mathcal{H}^{n-1}$ is an upper bound on the norm $\|u\|_{W^{1,p}(\partial B(0,\rho))}$ as defined in Section 5.3 analogously to (5.13). To this end suppose $\{(O^j, \varphi^j, \zeta^j)\}_{j\in\{1,\dots,N\}}$ is a fixed system of local maps and partition of unity for $\Omega = B(0,\rho)$. Without loss of generality we can assume the additional property that $\varphi^j \in C^\infty(\overline{O^j})$ and $(\varphi^j)^{-1} \in C^\infty(\overline{Q^n})$ for $j \in \{1,\dots,N\}$, so that there exists a constant $c > 0$ depending on $\{(O^j, \varphi^j, \zeta^j)\}_{j\in\{1,\dots,N\}}$ and therefore actually on $B(0,\rho)$ with

$$J^{(n-1)} D \left((\varphi^j)^{-1}\right)(x', 0) \ge c \tag{6.3}$$

for all $x' \in Q^{n-1} \subset \mathbb{R}^{n-1}$ and $j \in \mathbb{N}$, where $J^{(n-1)}$ stands for the $(n-1)$-dimensional Jacobian defined in Definition 5.11. As usual, we use the notation $x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. Next we assert that for every $j \in \{1,\dots,N\}$ and $h : \mathbb{R}^n \to [0,\infty)$ measurable it holds

$$\int_{Q^{n-1}} h\big((\varphi^j)^{-1}(x', 0)\big)\,\mathrm{d}x' \le \frac{1}{c} \int_{O^j \cap \partial B(0,\rho)} h(x)\,\mathrm{d}\mathcal{H}^{n-1}(x). \tag{6.4}$$

This can be derived from Proposition 5.12 in combination with (6.3). Thereto take $E = Q^{n-1} \cong Q^n \cap \{x_n = 0\} \subset \mathbb{R}^{n-1}$, $g(x') = h\big((\varphi^j)^{-1}(x', 0)\big)$ for $x' \in Q^{n-1}$ and $f$ the whole-space Lipschitz extension to $\mathbb{R}^{n-1}$ of $(\varphi^j)^{-1}$ restricted to $\{x_n = 0\}$ and observe that with this choice

$$\sum_{x' \in E \cap f^{-1}\{x\}} g(x') = \chi_{(O^j \cap \partial B(0,\rho))}(x)\, g(\varphi^j(x)) = \chi_{(O^j \cap \partial B(0,\rho))}(x) h(x)$$

for all $x \in \mathbb{R}^n$. Then, with $\varphi^j_*$ as in (5.11) estimate (6.4) yields

$$\begin{aligned}\sum_{j=1}^N \|\varphi^j_*(\zeta^j u)\|^p_{L^p(\mathbb{R}^{n-1})} &= \sum_{j=1}^N \int_{Q^{n-1}} \left|(\zeta^j u)\big((\varphi^j)^{-1}(x', 0)\big)\right|^p \mathrm{d}x' \\ &\le C \sum_{j=1}^N \int_{O^j \cap \partial B(0,\rho)} (\zeta^j(x))^p |u(x)|^p \,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &\le C \int_{\partial B(0,\rho)} \Big(\sum_{j=1}^N \zeta^j(x)\Big)^p |u(x)|^p \,\mathrm{d}\mathcal{H}^{n-1}(x) = C \int_{\partial B(0,\rho)} |u(x)|^p \,\mathrm{d}\mathcal{H}^{n-1}(x),\end{aligned}$$

and

$$
\begin{aligned}
\sum_{j=1}^{N} \|\nabla\big(\varphi_*^j(\zeta^j u)\big)\|_{L^p(\mathbb{R}^{n-1};\mathbb{R}^{n-1})}^p &= \sum_{j=1}^{N} \int_{Q^{n-1}} \Big(\sum_{i=1}^{n-1} \big|\partial_i(\zeta^j u)\big((\varphi^j)^{-1}(x',0)\big)\big|^2\Big)^{p/2} \,\mathrm{d}x' \\
&\le C \sum_{j=1}^{N} \int_{Q^{n-1}} \big|u\big((\varphi^j)^{-1}(x',0)\big)\big|^p + \big|\zeta^j\big((\varphi^j)^{-1}(x',0)\big)\big|^p \big|\nabla u\big((\varphi^j)^{-1}(x',0)\big)\big|^p \,\mathrm{d}x' \\
&\le C \sum_{j=1}^{N} \int_{O^j \cap \partial B(0,\rho)} |u(x)|^p + |\zeta^j(x)|^p |\nabla u(x)|^p \,\mathrm{d}\mathcal{H}^{n-1}(x) \\
&\le C \int_{\partial B(0,\rho)} |u(x)|^p + |\nabla u(x)|^p \,\mathrm{d}\mathcal{H}^{n-1}(x)
\end{aligned}
$$

with $C = C(p, n, B(0,\rho))$. In view of (5.13) this proves $\|u\|_{W^{1,p}(\partial B(0,\rho))} \le C \int_{\partial B(0,\rho)} |u|^p + |\nabla u|^p \,\mathrm{d}\mathcal{H}^{n-1}$. Finally, (6.1) gives the asserted estimate. □

**Remark 6.2** *By Remark 5.17 the result of Lemma 6.1 extends naturally to vector-valued functions.*

Next we present a lemma which implies the existence of an extension operator with enhanced integrability on functions in Sobolev boundary spaces. In this context the embedding and trace theorems of Section 5.3 will become valuable for proving higher integrability of the harmonic extension.

**Lemma 6.3** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^\infty$ boundary and $n \ge 2$. Further let $u \in W^{1,p}(\partial\Omega;\mathbb{R}^m)$ with $p \ge 2$. If $v$ denotes the harmonic extension of $u$ to $\Omega$, it holds*

$$\|v\|_{W^{1,np/(n-1)}(\Omega;\mathbb{R}^m)} \le c\,\|u\|_{W^{1,p}(\partial\Omega;\mathbb{R}^m)}. \tag{6.5}$$

*with a constant $c$ not depending on $u$.*

*PROOF.* Let $B^s_{p,q}(\Omega;\mathbb{R}^m)$, $H^s_p(\Omega;\mathbb{R}^m)$ and $W^s_p(\Omega;\mathbb{R}^m)$ be the Besov, Bessel-potential and Sobolev-Slobodeckij spaces introduced in Definition 5.13 under consideration of Remark 5.17. Since the corresponding boundary spaces for smooth domains are defined via charts as done in (5.13), we infer $W^{1,p}(\partial\Omega;\mathbb{R}^m) = W^1_p(\partial\Omega;\mathbb{R}^m) = H^1_p(\partial\Omega;\mathbb{R}^m)$ from Definition 5.13 (iii). By Theorem 5.15 (i) the embedding

$$H^1_p(\mathbb{R}^n) \hookrightarrow B^1_{p,p}(\mathbb{R}^n) \tag{6.6}$$

is continuous, so that $H^1_p(\partial\Omega;\mathbb{R}^m) \hookrightarrow B^1_{p,p}(\partial\Omega;\mathbb{R}^m)$ is continuous as well. Consequently, $u \in B^1_{p,p}(\partial\Omega;\mathbb{R}^m)$. Theorem 5.16 applied with $s = 1 + 1/p$ provides the existence of a continuous trace operator

$$\gamma : B^{1+1/p}_{p,p}(\Omega;\mathbb{R}^m) \to B^1_{p,p}(\partial\Omega;\mathbb{R}^m) \tag{6.7}$$

with continuous right inverse. Thus, there exists a $g \in B^{1+1/p}_{p,p}(\Omega;\mathbb{R}^m)$ with $\gamma(g) = g|_{\partial\Omega} = u$. Next we exploit the identity $B^{1+1/p}_{p,p}(\Omega;\mathbb{R}^m) = W^{1+1/p}_p(\Omega;\mathbb{R}^m)$, which is an immediate consequence of Definition 5.13 (iii), leading to $g \in W^{1+1/p}_p(\Omega;\mathbb{R}^m)$. By Theorem 5.15 (ii) applied for $s = 1 + 1/p$, $t = 1$ and $q = \frac{np}{(n-1)}$ we have the continuous embedding

$$W^{1+1/p}_p(\Omega;\mathbb{R}^m) \hookrightarrow W^{1,np/(n-1)}(\Omega;\mathbb{R}^m) \tag{6.8}$$

and thus, $g \in W^{1,np/(n-1)}(\Omega;\mathbb{R}^m)$. All in all, $u$ is the trace of a $W^{1,np/(n-1)}$-function. Recall that $v$ was the harmonic extension of $u$ on $\Omega$. By setting $w = v - g$ and rewriting the homogeneous Dirichlet problem for $v$ as an inhomogeneous boundary value problem with zero boundary data for $w$ we have

$$\begin{aligned} \Delta w &= \operatorname{div} G && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $G = -\nabla g \in L^{np/(n-1)}(\Omega;\mathbb{R}^{m\times n})$. At this point we make use of classical $L^p$-theory, see Theorem 6.9, to obtain

$$\|\nabla w\|_{L^{np/(n-1)}(\Omega;\mathbb{R}^m)} \le c\,\|G\|_{L^{np/(n-1)}(\Omega;\mathbb{R}^{m\times n})} = c\,\|\nabla g\|_{L^{np/(n-1)}(\Omega;\mathbb{R}^{m\times n})} \tag{6.9}$$

with the constant $c$ depending on $p$, $n$, $m$ and $\Omega$. Eventually, we join the all foregoing findings including (6.7), (6.8), (6.9) together and infer with the help of Poincaré's and the triangle inequality that

$$\begin{aligned} \|v\|_{W^{1,np/(n-1)}(\Omega;\mathbb{R}^m)} &\le c\,\|g\|_{W^{1,np/(n-1)}(\Omega;\mathbb{R}^m)} \le c\,\|g\|_{W^{1+1/p}_p(\Omega;\mathbb{R}^m)} \\ &= c\,\|g\|_{B^{1+1/p}_{p,p}(\Omega;\mathbb{R}^m)} \le c\,\|g\|_{B^1_{p,p}(\partial\Omega;\mathbb{R}^m)} \le c\,\|u\|_{W^{1,p}(\partial\Omega;\mathbb{R}^m)}. \end{aligned}$$

Hence, (6.5) is proven. □

## 6.2. Some results on anisotropic Sobolev functions

As we will see from the algebraic estimates in Section 7.1.1, the energy densities of (4.17) show highly anisotropic growth behavior. Therefore it is worth analyzing functions whose gradients display different properties of integrability depending on the directions considered. Actually, this paragraph is concerned with anisotropic Sobolev functions with one specific direction of higher integrability.
The following lemma is an approximation result for anisotropic Sobolev functions attained via convolution.

**Lemma 6.4** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $1 \le p < n$ and $q = \frac{np}{n-p}$. For every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ with $\partial_1 u \in L^q(\Omega;\mathbb{R}^m)$ and every $\mu > 0$ there exists a $v \in C^\infty(\Omega;\mathbb{R}^m)$ such that* *(i)* $\|u-v\|_{W^{1,p}(\Omega;\mathbb{R}^m)} \leq \mu$,

*(ii)* $\|\partial_1 u - \partial_1 v\|_{L^q(\Omega;\mathbb{R}^m)} \leq \mu$.

*PROOF.* This lemma follows from standard convolution techniques applied to a suitable decomposition of the set $\Omega$, see e.g. [31, Section 5.3].
We write $\Omega = \cup_{k=1}^{\infty}\Omega^k$, where $\Omega^k = \{x \in \Omega \mid \operatorname{dist}(x,\partial\Omega) > 1/k\}$, define $V^k = \Omega^{k+3} \setminus \overline{\Omega^{k+1}}$ for $k \in \mathbb{N}$ and choose $V^0 \subset\subset \Omega$ such that $\Omega \subset \cup_{k=0}^{\infty} V^k$. By $\{\xi^k\}_{k=0}^{\infty}$ we denote a smooth partition of unity with respect to $\{V^k\}_{k=0}^{\infty}$, meaning $0 \leq \xi^k \leq 1$, $\xi^k \in C_0^{\infty}(V^k), \sum_{k=0}^{\infty}\xi^k \equiv 1$ on $\Omega$. Now fix $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ with $\partial_1 u \in L^q(\Omega;\mathbb{R}^m)$. According to [31, Section 5.2, Theorem 1 (iv)] it holds $\xi^k u \in W^{1,p}(\Omega;\mathbb{R}^m)$ for every $k \in \mathbb{N}_0$ and we infer from the product rule that

$$\begin{aligned}\int_\Omega |\partial_1(\xi^k u)|^q \,\mathrm{d}x &\leq 2^{q-1}\left(\int_\Omega |\xi^k \partial_1 u|^q \,\mathrm{d}x + \int_\Omega |(u\otimes\nabla\xi^k)e_1|^q \,\mathrm{d}x\right)\\ &\leq 2^{q-1}\left(\|\partial_1 u\|^q_{L^q(\Omega;\mathbb{R}^m)} + \|\nabla\xi^k\|_{L^\infty(\Omega;\mathbb{R}^m)}\|u\|^q_{L^q(\Omega;\mathbb{R}^m)}\right) < c(k) < \infty.\end{aligned}$$

In the last step we used the continuous embedding $W^{1,p}(\Omega;\mathbb{R}^m) \hookrightarrow L^q(\Omega;\mathbb{R}^m)$ to find an upper bound on the expression. Note, however, that the constant $c$ still depends on $k$, for the gradients of $\xi^k$ are not uniformly bounded in $L^\infty$. All in all it has been shown that $\partial_1(\xi^k u) \in L^q(\Omega;\mathbb{R}^m)$ for each $k \in \mathbb{N}_0$.
Let us fix $\mu > 0$. For $k \in \mathbb{N}$ we define $W^k = \Omega^{k+4} \setminus \overline{\Omega^k}$ and take $W^0 \subset\subset \Omega$ with $V^0 \subset\subset W^0$. In the following, $\eta \in C^\infty(\mathbb{R}^n)$ denotes the standard mollifier and $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$ for $x \in \mathbb{R}^n$ and $\varepsilon > 0$. By choosing $\varepsilon_k > 0$ sufficiently small the convolution of $\xi^k u$ with $\eta_{\varepsilon_k}$, that is $v^k := \eta_{\varepsilon_k} * (\xi^k u) \in C^\infty(\Omega;\mathbb{R}^m)$, satisfies

(a) $\|v^k - \xi^k u\|_{W^{1,p}(\Omega;\mathbb{R}^m)} \leq \frac{\mu}{2^{k+1}}$,

(b) $\|\partial_1 v^k - \partial_1(\xi^k u)\|_{L^q(\Omega;\mathbb{R}^m)} \leq \frac{\mu}{2^{k+1}}$,

(c) $\operatorname{supp} v^k \subset W^k$

for all $k \in \mathbb{N}_0$. The points (a) and (c) are standard properties of mollifiers, see e.g. [31, Appendix C.4]. Regarding (b) we argue that $\partial_1 v^k = \partial_1[\eta_{\varepsilon_k} * (\xi^k u)] = \eta_{\varepsilon_k} * \partial_1(\xi^k u)$. In view of $\eta_{\varepsilon_k} * \partial_1(\xi^k u) \to \partial_1(\xi^k u)$ in $L^q(\Omega;\mathbb{R}^m)$ as $\varepsilon \to 0$ the expression in (b) can be made small enough by an appropriate choice of $\varepsilon_k$.
Eventually, we set $v = \sum_{k=0}^{\infty} v^k$, which is well-defined accounting for (c). For every $V \subset\subset \Omega$ the sum $\sum_{k=0}^{\infty} v^k$ has at most finitely many non-zero terms, so that $v^k \in C^\infty(W^k;\mathbb{R}^m)$ for all $k \in \mathbb{N}_0$ results in $v \in C^\infty(\Omega;\mathbb{R}^m)$. Since $u = \sum_{k=0}^{\infty}\xi^k u$, (a) and (b) yield for each $V \subset\subset \Omega$ that

$$\|v-u\|_{W^{1,p}(V;\mathbb{R}^m)} \leq \sum_{k=0}^{\infty}\|v^k - \xi^k u\|_{W^{1,p}(\Omega;\mathbb{R}^m)} \leq \sum_{k=0}^{\infty}\frac{\mu}{2^{k+1}} = \mu,$$

and similarly,

$$\|\partial_1 v - \partial_1 u\|_{L^q(V;\mathbb{R}^m)} \leq \sum_{k=0}^{\infty} \|\partial_1 v^k - \partial_1(\xi^k u)\|_{L^q(\Omega;\mathbb{R}^m)} \leq \sum_{k=0}^{\infty} \frac{\mu}{2^{k+1}} = \mu.$$

The assertion results from taking the supremum over all $V \subset\subset \Omega$. □

With this at hand one can prove a kind of Sobolev embedding for anisotropic functions in a two-dimensional setting.

**Proposition 6.5** *Let $\Omega \subset \mathbb{R}^2$ be open and bounded and $1 < q < \infty$. Suppose $u \in W^{1,1}(\Omega;\mathbb{R}^m)$ with $\partial_1 u \in L^q(\Omega;\mathbb{R}^m)$. Then, $u \in L^{2q}(\Omega;\mathbb{R}^m)$ and there is the estimate*

$$\|u\|_{L^{2q}(\Omega;\mathbb{R}^m)} \leq c\left(\|u\|_{W^{1,1}(\Omega;\mathbb{R}^m)} + \|\partial_1 u\|_{L^q(\Omega;\mathbb{R}^m)}\right) \tag{6.10}$$

*with $c > 0$ depending on $q$, $\Omega$ and $m$.*

*PROOF.* We will only present here an explicit proof for real-valued functions. The vector-valued case is an immediate consequence of applying the real-valued version to each of the components. A second simplification is that we restrict the proof to the situation that $\Omega \subset \mathbb{R}^2$ is the unit square. The general claim can then be derived by covering $\Omega$ with squares using Vitali's covering theorem.
Assume for the moment that $u \in C^1(\Omega)$ with $\Omega = (0,1)^2$. According to the fundamental theorem of calculus it holds for $x = (x_1, x_2)$ and $(x_1, \tilde{x}_2) \in \Omega$ that $u(x_1,x_2) - u(x_1,\tilde{x}_2) = \int_{\tilde{x}_2}^{x_2} \partial_2 u(x_1,t)\,\mathrm{d}t$, which implies $|u(x_1,x_2)| \leq |u(x_1,\tilde{x}_2)| + \int_0^1 |\partial_2 u(x_1,t)|\,\mathrm{d}t$. Integration with respect to $\tilde{x}_2$ yields

$$|u(x)| = |u(x_1,x_2)| \leq \int_0^1 |u(x_1,t)|\,\mathrm{d}t + \int_0^1 |\partial_2 u(x_1,t)|\,\mathrm{d}t. \tag{6.11}$$

With analogous considerations, exchanging the roles of $x_1$ and $x_2$ and taking $u^{2q-1}$ instead of $u$,

$$|u(x)|^{2q-1} \leq \int_0^1 |u(s,x_2)|^{2q-1}\,\mathrm{d}s + (2q-1)\int_0^1 |u(s,x_2)|^{2q-2}|\partial_1 u(s,x_2)|\,\mathrm{d}s. \tag{6.12}$$

Then, multiplication of (6.11) and (6.12) together with Fubini's theorem and Hölder's inequality provide

$$\begin{aligned}
\|u\|_{L^{2q}(\Omega)}^{2q} &= \int_\Omega |u(x)|\,|u(x)|^{2q-1}\,\mathrm{d}x \\
&\leq \left(\int_\Omega |u(x)|\,\mathrm{d}x + \int_\Omega |\,\partial_2 u(x)|\,\mathrm{d}x\right) \\
&\qquad \cdot \left(\int_\Omega |u(x)|^{2q-1}\,\mathrm{d}x + (2q-1)\int_\Omega |u(x)|^{2q-2}|\partial_1 u(x)|\,\mathrm{d}x\right) \\
&\leq \|u\|_{W^{1,1}(\Omega)}\left(\|u\|_{L^{2q}(\Omega)}^{2q-1} + (2q-1)\|\partial_1 u\|_{L^q(\Omega)}\|u\|_{L^{2q}(\Omega)}^{2q-2}\right).
\end{aligned}$$

Hence, $\|u\|^2_{L^{2q}(\Omega)} \leq \|u\|_{W^{1,1}(\Omega)} \left(\|u\|_{L^{2q}(\Omega)} + (2q-1)\|\partial_1 u\|_{L^q(\Omega)}\right)$, which becomes

$$\begin{aligned} \|u\|_{L^{2q}(\Omega)} &\leq 2\|u\|_{W^{1,1}(\Omega)} + 2\sqrt{2q-1}\|u\|^{1/2}_{W^{1,1}(\Omega)}\|\partial_1 u\|^{1/2}_{L^q(\Omega)} \\ &\leq (2+\sqrt{2q-1})\left(\|u\|_{W^{1,1}(\Omega)} + \|\partial_1 u\|_{L^q(\Omega)}\right) \end{aligned} \tag{6.13}$$

by algebraic calculations.
An argumentation via approximation will finally extend the previous findings to $u \in W^{1,1}(\Omega)$ with $\partial_1 u \in L^q(\Omega)$. By Lemma 6.4 there exists for every $k \in \mathbb{N}$ a smooth function $v^k \in C^\infty(\Omega)$ such that $\|v^k - u\|_{W^{1,1}(\Omega)} \leq \frac{1}{k}$ and $\|\partial_1 v^k - \partial_1 u\|_{L^q(\Omega)} \leq \frac{1}{k}$. Therefore, by applying the line of arguments resulting in (6.13) to each $v^k$ with $k \in \mathbb{N}$ one obtains

$$\begin{aligned} \|v^k\|_{L^{2q}(\Omega)} &\leq (2+\sqrt{2q-1})\left(\|v^k\|_{W^{1,1}(\Omega)} + \|\partial_1 v^k\|_{L^q(\Omega)}\right) \\ &\leq (2+\sqrt{2q-1})\left(\|u\|_{W^{1,1}(\Omega)} + \frac{1}{k} + \|\partial_1 u\|_{L^q(\Omega)} + \frac{1}{k}\right). \end{aligned}$$

Since the right-hand side of the upper inequality is uniformly bounded and $v^k \to u$ in $L^1(\Omega)$ for $k \to \infty$, we infer $v^k \rightharpoonup u$ in $L^{2q}(\Omega)$ as $k$ tends to $\infty$. By the weak lower semicontinuity of the $L^{2q}$-norm we get

$$\|u\|_{L^{2q}(\Omega)} \leq \liminf_{k\to\infty} \|v^k\|_{L^{2q}(\Omega)} = (2+\sqrt{2q-1})\left(\|u\|_{W^{1,1}(\Omega)} + \|\partial_1 u\|_{L^q(\Omega)}\right),$$

which is exactly the stated estimate. □

The next lemma deals with the question of equality between $\det \nabla u$ and the distributional determinant $\mathrm{Det}\, \nabla u$ for elements $u$ of a special type of anisotropic Sobolev space. Before we start to formulate the result in detail, let us give some general information on these two notions of determinant. See also [61] and the references therein.
In the two-dimensional setting, that is for functions $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, the Jacobian determinant of $u$ reads

$$\det \nabla u = \partial_1 u_1\, \partial_2 u_2 - \partial_2 u_1\, \partial_1 u_2,$$

and $\mathrm{Det}\, \nabla u$ is defined by

$$\mathrm{Det}\, \nabla u = \partial_2(u_2\, \partial_1 u_1) - \partial_1(u_2\, \partial_2 u_1) \quad \text{in} \quad \mathcal{D}'(\Omega),$$

meaning

$$\langle \mathrm{Det}\, \nabla u, \varphi\rangle = \langle \mathrm{Det}\, \nabla u, \varphi\rangle_{\mathcal{D}'(\Omega), C_0^\infty(\Omega)} = \int_\Omega u_2\, \partial_2 u_1\, \partial_1 \varphi - u_2\, \partial_1 u_1\, \partial_2 \varphi \,\mathrm{d}x$$

for all $\varphi \in C_0^\infty(\Omega)$. Note that by Sobolev's embedding theorem and Hölder's inequality $\mathrm{Det}\, \nabla u$ is well-defined for $u \in W^{1,4/3}(\Omega; \mathbb{R}^2)$. If $u \in C^2(\Omega; \mathbb{R}^2)$ one has the classical

equality Det $\nabla u = \det \nabla u$ following immediately from integration by parts. It says that $\det \nabla u$ can be written as a divergence. Besides, there is the well-known result that $\det \nabla u$ and Det $\nabla u$ are equal in the sense of distributions for $u \in W^{1,2}(\Omega;\mathbb{R}^2)$, which can be shown via approximation with smooth functions. In general, however, Det $\nabla u = \det \nabla u$ is not true. This can be seen from the famous counterexample $u : B(0,1) \subset \mathbb{R}^2 \to \mathbb{R}^2$, $u(x) = \frac{x}{|x|}$, where $\det \nabla u = 0$ almost everywhere, while Det $\nabla u = \pi\delta_0$ with $\delta_0$ the Dirac mass in 0.
For later calculations it will be convenient to rewrite Det $\nabla u$ in the form

$$\langle \text{Det } \nabla u, \varphi\rangle = \int_\Omega u_2(\nabla u_1 \cdot J\nabla\varphi)\,\mathrm{d}x = \int_\Omega (u\cdot e_2)\Big(\nabla(u\cdot e_1)\cdot J\nabla\varphi\Big)\,\mathrm{d}x \tag{6.14}$$

with $\varphi \in C_0^\infty(\Omega)$, $e_1, e_2$ the standard unit vectors in $\mathbb{R}^2$ and $J = e_2 \otimes e_1 - e_1 \otimes e_2$ the counterclockwise rotation by $\pi/2$ in the plane.

**Lemma 6.6** *Let $\Omega \subset \mathbb{R}^2$ be open and bounded, $p \in (1, 4/3]$ and $q = \frac{p}{p-1}$. Assume $u \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$ with $\partial_1 u \in L^q(\Omega;\mathbb{R}^2)$. Then,* Det $\nabla u$ *is well-defined in the sense of distributions and*

$$\det \nabla u = \text{Det } \nabla u \qquad \textit{in } \mathcal{D}'(\Omega),$$

*meaning* $\int_\Omega \det \nabla u\, \varphi\,\mathrm{d}x = \langle \text{Det } \nabla u, \varphi\rangle$ *for all* $\varphi \in C_0^\infty(\Omega)$.

*PROOF.* Let $\varphi \in C_0^\infty(\Omega)$ be an arbitrary test function. By Lemma 6.4 and the standard properties of mollifiers we can approximate $u$ for every $k \in \mathbb{N}$ by $v^k \in C^\infty(\Omega;\mathbb{R}^2)$, so that

$$\|v^k - u\|_{W^{1,p}(\Omega;\mathbb{R}^2)} \le \frac{1}{k}, \quad \|v^k - u\|_{L^q(\Omega;\mathbb{R}^2)} \le \frac{1}{k}, \quad \|\partial_1 v^k - \partial_1 u\|_{L^q(\Omega;\mathbb{R}^2)} \le \frac{1}{k}. \tag{6.15}$$

Consider next the limit $k \to \infty$. Our goal is to show

$$\lim_{k\to\infty} \langle \text{Det } \nabla v^k, \varphi\rangle = \langle \text{Det } \nabla u, \varphi\rangle, \tag{6.16}$$

$$\lim_{k\to\infty} \int_\Omega \det v^k\, \varphi\,\mathrm{d}x = \int_\Omega \det \nabla u\,\varphi\,\mathrm{d}x. \tag{6.17}$$

Based on (6.15) and (6.14) there is the estimate

$$\begin{aligned}
&\big|\langle \text{Det } \nabla v^k - \text{Det } \nabla u, \varphi\rangle\big| \\
&\le \int_\Omega \big|(v^k - u)\cdot e_2\big|\,\big|\nabla(v^k\cdot e_1)\big|\,\big|J\nabla\varphi\big|\,\mathrm{d}x \\
&\qquad\qquad + \int_\Omega \big|u\cdot e_2\big|\,\big|\nabla(v^k\cdot e_1) - \nabla(u\cdot e_1)\big|\,\big|J\nabla\varphi\big|\,\mathrm{d}x \\
&\le \|\nabla\varphi\|_{L^\infty(\Omega;\mathbb{R}^2)}\big(\|v^k - u\|_{L^q(\Omega;\mathbb{R}^2)}\|v^k\|_{W^{1,p}(\Omega;\mathbb{R}^2)} + \|u\|_{L^q(\Omega;\mathbb{R}^2)}\|v^k - u\|_{W^{1,p}(\Omega;\mathbb{R}^2)}\big) \\
&\le \|\nabla\varphi\|_{L^\infty(\Omega;\mathbb{R}^2)}\big(\|u\|_{L^q(\Omega;\mathbb{R}^2)} + \|u\|_{W^{1,p}(\Omega;\mathbb{R}^2)} + 1\big)\,\frac{1}{k}
\end{aligned}$$

for all $k \in \mathbb{N}$, which proves (6.16). The proof of (6.17) again rests upon the fact that $p$ and $q$ are dual exponents. Thus, (6.17) is a consequence of

$$\begin{aligned}\Big|\int_\Omega \big(\det \nabla v^k - \det \nabla u\big)\varphi \,\mathrm{d}x\Big| &\le \int_\Omega \big|(\partial_1 v^k \cdot J\partial_2 v^k) - (\partial_1 u \cdot J\partial_2 u)\big||\varphi| \,\mathrm{d}x \\ &\le \|\varphi\|_{L^\infty(\Omega)}\big(\|\partial_1 v^k\|_{L^q(\Omega;\mathbb{R}^2)}\|v^k - u\|_{W^{1,p}(\Omega;\mathbb{R}^2)} + \|\partial_1 v^k - \partial_1 u\|_{L^q(\Omega;\mathbb{R}^2)}\|u\|_{W^{1,p}(\Omega;\mathbb{R}^2)}\big) \\ &\le \|\varphi\|_{L^\infty(\Omega)}\big(\|\partial_1 u\|_{L^q(\Omega;\mathbb{R}^2)} + \|u\|_{W^{1,p}(\Omega;\mathbb{R}^2)} + 1\big)\frac{1}{k}\end{aligned}$$

with $k \in \mathbb{N}$. Finally, we are in the position to compose all our findings and obtain

$$\langle \mathrm{Det}\ \nabla u,\ \varphi\rangle = \lim_{k\to\infty} \langle \mathrm{Det}\ \nabla v^k,\ \varphi\rangle = \lim_{k\to\infty}\int_\Omega \det \nabla v^k\ \varphi \,\mathrm{d}x = \int_\Omega \det \nabla u\ \varphi\,\mathrm{d}x.$$

The second equality is due to the equivalence of the notions $\det \nabla$ and $\mathrm{Det}\ \nabla$ on $C^2$-functions. Since $\varphi \in C_0^\infty(\Omega)$ was chosen arbitrarily, the claim is shown. □

At the end of this section let us state a probably well-known result. The proof is given here for the readers' convenience.

**Lemma 6.7** *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set, $1 < p < \infty$ and suppose there is a sequence $\{f^k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ such that $\|f^k\|_{L^p(\Omega;\mathbb{R}^m)} \le C$ for all $k \in \mathbb{N}$ and $f^k \to f$ in $L^1(\Omega;\mathbb{R}^m)$ as $k \to \infty$. Then, for $q \in (1,p)$ it holds*

$$f^k \to f \quad \text{in} \quad L^q(\Omega;\mathbb{R}^m) \qquad \text{as } k \to \infty.$$

*PROOF.* For $q \in (1,p)$ there is $t \in (0,1)$ such that $p = \frac{q-t}{1-t}$. By Hölder's inequality we find

$$\int_\Omega |f^k - f|^q \,\mathrm{d}x = \int_\Omega |f^k - f|^t|f^k - f|^{q-t}\,\mathrm{d}x \le \|f^k - f\|^t_{L^1(\Omega;\mathbb{R}^m)}\|f^k - f\|^{p(1-t)}_{L^p(\Omega;\mathbb{R}^m)}.$$

This shows the claim, since $\|f^k - f\|_{L^p(\Omega;\mathbb{R}^m)}$ is uniformly bounded and $\|f^k - f\|_{L^1(\Omega;\mathbb{R}^m)} \to 0$ for $k \to \infty$. □

## 6.3. $L^p$-theory for Dirichlet problems with focus on weak-$L^1$ estimates

This section recapitulates some standard results from $L^p$-theory in the context of Poisson's equation with Dirichlet boundary data and the divergence of an $L^p$-function as a right-hand side. Moreover, we derive weak-$L^1$ estimates in versions which are exactly adjusted to what we need in the subsequent sections, but which are not frequently covered in the literature. To make both arguments and notation easier we stick to real-valued

functions for this paragraph and keep in mind that the generalizations to vector-valued functions are immediate by reasoning componentwise.
Let us consider the Dirichlet problem

$$\begin{aligned} \Delta u &= \operatorname{div} f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{6.18}$$

with $f \in L^p(\Omega; \mathbb{R}^n)$ for $1 < p < \infty$. By the way, in this section we assume $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ to be a bounded domain with smooth boundary, say of class $C^\infty$.
Then the notion of a weak solution for (6.18) is as follows.

**Definition 6.8** *A function $u : \Omega \to \mathbb{R}$ is said to be a weak solution of* (6.18) *with $1 < p < \infty$, if $u \in W_0^{1,p}(\Omega)$ satisfies*

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f \cdot \nabla \varphi \, \mathrm{d}x$$

*for all $\varphi \in C_0^\infty(\Omega)$.*

By now there are different approaches of how to manage the task of developing an appropriate $L^p$-theory for elliptic equations and boundary value problems such as (6.18). In [71] generalizations of Garding's inequality and the classical Hilbert space arguments are derived so that one can follow the line of thoughts used in the context of $L^2$-spaces, whereas [76] and [39] use the concept of interpolation in combination with Schauder estimates. Another very common technique consists in investigating fundamental solutions and Poisson-kernels and in exploiting results from the field of singular integrals, especially the well-known Calderón-Zygmund theorem, which we stated in Theorem 5.23. As a reference for this approach see [40] and [1] among others.
Next we state the essential existence and regularity theorem for the Dirichlet problem under consideration, as it can be found for example in [39, Chapter 7] for $p \geq 2$. Then the statement for $p \in (1, 2)$ follows from duality arguments, which basically rely on applying [4, 10.5 Satz] to the weak Laplace operator on $W_0^{1,q}(\Omega)$, where $q = \frac{p}{p-1} > 2$.

**Theorem 6.9** *The problem* (6.18) *with $1 < p < \infty$ has a unique weak solution $u \in W_0^{1,p}(\Omega)$ and there is a constant $c = c(n, p, \Omega) > 0$ such that*

$$\|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)} \leq c \, \|f\|_{L^p(\Omega;\mathbb{R}^n)}.$$

Notice that $p = 1$ is not covered by the foregoing definition and theorem. Actually, this case requires a separate treatment and one cannot expect to achieve a result that is in perfect correspondence with Theorem 6.9. As a consequence of lacking compactness the associated estimates will only be of weak type in the sense of Section 5.5. Here we state a proposition providing weak-$L^1$ estimates for classical solutions of problem (6.18) and give a detailed proof to illustrate the use of singular integral methods.

**Proposition 6.10** *Suppose $f \in C^\infty(\overline{\Omega};\mathbb{R}^n)$. If $u$ is the classical solution of*

$$\begin{aligned} \Delta u &= \operatorname{div} f \qquad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \partial\Omega \end{aligned} \tag{6.19}$$

*and $V \subset\subset \Omega$, then there is the estimate*

$$\|\nabla u\|_{L^1_w(V;\mathbb{R}^n)} \le c\, \|f\|_{L^1(\Omega;\mathbb{R}^n)}$$

*with the constant $c > 0$ depending only on $V$ and $n$.*

Before we concentrate directly on the Dirichlet problem (6.19), let us discuss first the corresponding situation on the whole space. Be aware that throughout this paragraph we will be using the notation and the quantities from Section 5.4.

**Lemma 6.11** *Let $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n;\mathbb{R}^n)$ such that $\operatorname{supp} f \subset \mathbb{R}^n$ is compact. Then, the function $w : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$w(x) = -\int_{\mathbb{R}^n} \widetilde{\Phi}(x-y) \cdot f(y)\, \mathrm{d}y = -\sum_{j=1}^n \big(\widetilde{\Phi}_j * f_j\big)(x), \qquad x \in \mathbb{R}^n$$

*is a solution of $\Delta w = \operatorname{div} f$ in $\mathcal{D}'(\mathbb{R}^n)$. It holds $w \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ and*

$$\|\nabla w\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \le c\, \|f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \tag{6.20}$$

*with $c$ depending only on $n$ and $p$. Further, there is a constant $c = c(n)$ such that*

$$\|\nabla w\|_{L^1_w(\mathbb{R}^n;\mathbb{R}^n)} \le c\, \|f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}. \tag{6.21}$$

*PROOF.* From $\widetilde{\Phi} \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n;\mathbb{R}^n)$ with compact support we infer $w \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ using standard properties of convolution, see e.g. [5, Section 2.1]. So by Fubini's Theorem and (5.17) we obtain for all $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned} \langle \Delta w, \varphi \rangle &= \langle \Delta w, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n)} = \int_{\mathbb{R}^n} w(x) \Delta\varphi(x)\, \mathrm{d}x \\ &= -\sum_{j=1}^n \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \widetilde{\Phi}_j(x-y)\, f_j(y)\, \mathrm{d}y \right) \Delta\varphi(x)\, \mathrm{d}x \\ &= -\int_{\mathbb{R}^n} \sum_{j=1}^n \left( \int_{\mathbb{R}^n} \widetilde{\Phi}_j(z)\, \Delta\varphi(y+z)\, \mathrm{d}z \right) f_j(y)\, \mathrm{d}y \\ &= -\int_{\mathbb{R}^n} \nabla\varphi(y) \cdot f(y)\, \mathrm{d}y = \langle \operatorname{div} f, \varphi \rangle. \end{aligned}$$

In order to prove the estimates (6.20) and (6.21) we will use the theory of singular integrals introduced in Section 5.6. To this end we start by defining appropriate integral kernels. For $i, j \in \{1, \ldots, n\}$ let $K^{ij} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be given by

$$K^{ij}(x) = \partial_i \widetilde{\Phi}_j(x) = \frac{\omega^{ij}(x)}{|x|^n} \quad \text{with} \quad \omega^{ij}(x) = \frac{1}{n\omega_n}\Big(n\frac{x_i x_j}{|x|^2} - \delta_{ij}\Big),$$

where $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$. In view of $\int_{\partial B(0,1)} \omega^{ij}(x)\ \mathrm{d}\mathcal{H}^{n-1}(x) = \frac{1}{n\omega_n}\int_{\partial B(0,1)} (n x_i x_j - \delta_{ij})\ \mathrm{d}\mathcal{H}^{n-1}(x) = 0$ the function $K^{ij}$ clearly fulfills the requirements of a Calderón-Zygmund kernel according to Definition 5.22 for each pair $(i,j)$ with $i, j \in \{1, \ldots, n\}$. Besides, it is obvious that the restriction of $\omega^{ij}$ to $\partial B(0,1)$ is Lipschitz continuous, so that $K^{ij}$ satisfies all the conditions necessary for applying Theorem 5.23 and Proposition 5.24. Following the notation of Section 5.6 we write for $x \in \mathbb{R}^n$ and $i, j \in \{1, \ldots, n\}$

$$\begin{aligned} T^{ij}_\varepsilon(f_j)(x) &= \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K^{ij}(x-y) f_j(y)\ \mathrm{d}y, \\ T^{ij}_*(f_j)(x) &= \sup_{\varepsilon > 0} \big|T^{ij}_\varepsilon(f_j)(x)\big| \end{aligned}$$

and take the pointwise limit

$$T^{ij}(f_j)(x) \;=\; \int_{\mathbb{R}^n} K^{ij}(x-y) f_j(y)\ \mathrm{d}y = \lim_{\varepsilon \to 0} T^{ij}_\varepsilon(f_j)(x), \tag{6.22}$$

which exists for almost all $x \in \mathbb{R}^n$ by Proposition 5.24.
The next step is to find a relation between the singular integrals $T^{ij}(f_j)$ and the weak derivatives of $w$. With this in mind let us first have a look at the interaction between the approximating quantities $T^{ij}_\varepsilon(f_j)$ and $\partial_i w^j_\varepsilon$, where $w^j_\varepsilon$ is defined as $w^j_\varepsilon(x) = -\int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \widetilde{\Phi}_j(x-y) f_j(y)\ \mathrm{d}y$ for $x \in \mathbb{R}^n$. Using the theorem by Fubini-Tonelli and integration by parts yields

$$\begin{aligned} \langle \partial_i w^j_\varepsilon, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^n), C^\infty_0(\mathbb{R}^n)} &= -\int_{\mathbb{R}^n} w^j_\varepsilon(x) \partial_i \varphi(x)\ \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \Big( \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \widetilde{\Phi}_j(x-y) f_j(y)\ \mathrm{d}y \Big) \partial_i \varphi(x)\ \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \Big( \int_{\mathbb{R}^n \setminus B(y,\varepsilon)} \widetilde{\Phi}_j(x-y) \partial_i \varphi(x)\ \mathrm{d}x \Big) f_j(y)\ \mathrm{d}y \\ &= -\int_{\mathbb{R}^n} \Big( \int_{\mathbb{R}^n \setminus B(y,\varepsilon)} K^{ij}(x-y) \varphi(x)\ \mathrm{d}x \\ &\qquad\qquad + \int_{\partial B(y,\varepsilon)} \widetilde{\Phi}_j(x-y) \varphi(x) \nu_i(x)\ \mathrm{d}\mathcal{H}^{n-1}(x) \Big) f_j(y)\ \mathrm{d}y \end{aligned}$$

$$= -\int_{\mathbb{R}^n} T_\varepsilon^{ij}(f_j)(x)\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}^n}\Big(\int_{\partial B(y,\varepsilon)} \widetilde{\Phi}_j(x-y)\varphi(x)\nu_i(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\Big) f_j(y)\,\mathrm{d}y \quad (6.23)$$

for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and all $i,j \in \{1,\dots,n\}$. Here $\nu$ stands for the unit outer normal on $\partial B(y,\varepsilon)$, which is the mapping $\nu : \partial B(y,\varepsilon) \to \mathbb{R}^n$ given by $\nu(x) = \frac{x-y}{|x-y|}$. Note that the order of integration can be permuted in this context using Fubini-Tonelli, because both $\widetilde{\Phi}_j(\,.-y)\chi_{\mathbb{R}^n\setminus B(y,\varepsilon)}(.)$ and $K^{ij}(\,.-y)\chi_{\mathbb{R}^n\setminus B(y,\varepsilon)}(.)$ are bounded on compact sets for every $\varepsilon > 0$ and all $y \in \mathbb{R}^n$.
As a next step one has to pass to the limit in (6.23). Notice that $\{w_\varepsilon^j\}_{\varepsilon>0}$ is bounded pointwise by the Riesz-potential $I_1(|f_j|)$, see Definition 5.25, up to a constant depending only on $n$. By the Hardy-Littlewood-Sobolev inequality from Theorem 5.26 the expression $I_1(|f_j|)$ is locally integrable on $\mathbb{R}^n$. Further, we have $w_\varepsilon^j \to w^j = -\int_{\mathbb{R}^n}\widetilde{\Phi}_j(\,.-y)f_j(y)\,\mathrm{d}y$ pointwise almost everywhere in $\mathbb{R}^n$. So we conclude from Lebegue's theorem and the definition of distributional derivatives that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\langle \partial_i w_\varepsilon^j, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^n),C_0^\infty(\mathbb{R}^n)} &= -\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} w_\varepsilon^j(x)\partial_i\varphi(x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^n}\big(\lim_{\varepsilon\to 0} w_\varepsilon^j(x)\big)\partial_i\varphi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} w^j(x)\partial_i\varphi(x)\,\mathrm{d}x \\
&= \langle \partial_i w^j, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^n),C_0^\infty(\mathbb{R}^n)}
\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, which shows

$$\partial_i w_\varepsilon^j \to \partial_i w^j \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \qquad \text{as } \varepsilon \to 0 \tag{6.24}$$

for $i,j \in \{1,\dots,n\}$. In order to handle the term $\int_{\mathbb{R}^n} T_\varepsilon^{ij}(f_j)(x)\varphi(x)\,\mathrm{d}x$ in (6.23) we exploit that $f \in L^p(\mathbb{R}^n;\mathbb{R}^n)$ with compact support by assumption, so that Proposition 5.24 yields for every $V \subset\subset \mathbb{R}^n$ and all $i,j \in \{1,\dots,n\}$,

$$\|T_*^{ij}(f_j)\|_{L^1(V)} \le c\,\|T_*^{ij}(f_j)\|_{L^p(V)} \le c\,\|T_*^{ij}(f_j)\|_{L^p(\mathbb{R}^n)} \le c\,\|f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} < \infty$$

with a constant $c = c(p,n,V)$. Consequently, the sequence $\{T_\varepsilon^{ij}(f_j)\chi_V\}_{\varepsilon>0}$ has an integrable pointwise upper bound for every $V \subset\subset \mathbb{R}^n$. Hence, for $\varphi \in C_0^\infty(\Omega)$ one obtains by setting $V = \operatorname{supp}\varphi$ and applying Lebesgue's convergence theorem that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} T_\varepsilon^{ij}(f_j)(x)\varphi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n}\big(\lim_{\varepsilon\to 0} T_\varepsilon^{ij}(f_j)(x)\big)\varphi(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} T^{ij}(f_j)(x)\varphi(x)\,\mathrm{d}x. \qquad (6.25)
\end{aligned}$$

The last equality simply uses the definition of $T^{ij}(f_j)$ as the pointwise limit of $T_\varepsilon^{ij}(f_j)$ for $\varepsilon \to 0$, see (6.22).

As far as the boundary term in (6.23) is concerned we find by explicit calculation that

$$\begin{aligned}\lim_{\varepsilon\to 0}&\int_{\partial B(y,\varepsilon)} \widetilde{\Phi}_j(x-y)\varphi(x)\nu_i(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\\ &= \frac{1}{n\omega_n}\lim_{\varepsilon\to 0}\int_{\partial B(y,\varepsilon)} \frac{(y_i-x_i)(y_j-x_j)}{\varepsilon^{n+1}}\varphi(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\\ &= \frac{1}{n\omega_n}\lim_{\varepsilon\to 0}\int_{\partial B(0,1)} \varphi(\varepsilon z+y)z_iz_j\,\mathrm{d}\mathcal{H}^{n-1}(z) = c^{ij}\varphi(y),\end{aligned}$$

where $c^{ij}=c^{ij}(n)=\frac{1}{n\omega_n}\int_{\partial B(0,1)} z_iz_j\,\mathrm{d}\mathcal{H}^{n-1}(z)$ and $i,j\in\{1,\dots,n\}$. Since

$$\int_{\partial B(y,\varepsilon)} |\widetilde{\Phi}(x-y)|\,\mathrm{d}\mathcal{H}^{n-1}(x) = \frac{1}{n\omega_n}\varepsilon^{-(n-1)}\mathcal{H}^{n-1}\big(\partial B(y,\varepsilon)\big) = \frac{1}{n\omega_n}\mathcal{H}^{n-1}\big(\partial B(0,1)\big) = 1$$

for all $\varepsilon>0$ and $y\in\mathbb{R}^n$, the expression $\int_{\partial B(y,\varepsilon)}\widetilde{\Phi}_j(x-y)\varphi(x)\nu_i(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)$ is uniformly bounded with respect to $\varepsilon$ and $y$. So it follows from Lebesgue's convergence theorem that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}&\Big(\int_{\partial B(y,\varepsilon)} \widetilde{\Phi}_j(x-y)\varphi(x)\nu_i(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\Big)f_j(y)\,\mathrm{d}y\\ &= c^{ij}\int_{\mathbb{R}^n} f_j(y)\varphi(y)\,\mathrm{d}y \qquad (6.26)\end{aligned}$$

for $i,j\in\{1,\dots,n\}$. Joining the limiting results of (6.24), (6.25) and (6.26) together in (6.23) eventually shows

$$\partial_i w^j = -T^{ij}(f_j) - c^{ij}f_j \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

In view of Theorem 5.23 (iii) we obtain

$$\|\nabla w\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \le c\|f\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \qquad (6.27)$$

with a constant $c=c(n,p)$, which is the stated estimate (6.20). Additionally, (6.27) implies $\partial_i w^j = -T^{ij}(f_j)-c^{ij}f_j$ almost everywhere in $\mathbb{R}^n$. So we infer from the weak $(1,1)$ estimate of Theorem 5.24,

$$\begin{aligned}\|\partial_i w^j\|_{L^1_w(\mathbb{R}^n)} &\le 2\,\big(\|T^{ij}(f_j)\|_{L^1_w(\mathbb{R}^n)} + |c^{ij}|\,\|f_j\|_{L^1_w(\mathbb{R}^n)}\big)\\ &\le 2\,\big(\|T^{ij}_*(f_j)\|_{L^1_w(\mathbb{R}^n)} + |c^{ij}|\,\|f_j\|_{L^1(\mathbb{R}^n)}\big) \le c\,\|f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}\end{aligned}$$

for all $i,j\in\{1,\dots,n\}$ with $c=c(n)$. As a consequence there is a constant $c$ depending only on $n$ with

$$\|\nabla w\|_{L^1_w(\mathbb{R}^n;\mathbb{R}^n)} \le c\,\|f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)},$$

as postulated. □

The next lemma deals with the corrector terms that have to be controlled in order to achieve results on bounded domains.

**Lemma 6.12** *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with $\partial\Omega$ of class $C^\infty$ and $f \in L^1(\Omega;\mathbb{R}^n)$. Further let $v : \Omega \to \mathbb{R}$ be given by*

$$v(x) = \int_\Omega \nabla\phi_x(y) \cdot f(y) \,\mathrm{d}y.$$

*Then, $v$ solves $\Delta v = 0$ in $\mathcal{D}'(\Omega)$ and for every $V \subset\subset \Omega$ there exists a constant $c = c(n,V)$ with*

$$\|\nabla v\|_{L^\infty(V;\mathbb{R}^n)} \leq c\, \|f\|_{L^1(\Omega;\mathbb{R}^n)}. \tag{6.28}$$

*PROOF.* Since all the derivatives of $\phi$ defined in (5.16) are uniformly bounded with respect to $V \times \Omega$ for every $V \subset\subset \Omega$ by the statements of Section 5.4 and since $f$ is integrable on $\Omega$ by assumption, it is immediate that $v \in L^1_{\mathrm{loc}}(\Omega)$. With the help of Fubini-Tonelli we infer from the fact that $\phi_x$ is a harmonic function on $\Omega$ for every $x \in \Omega$ that $v$ is harmonic on $\Omega$ as well. At length, for every $\varphi \in C_0^\infty(\Omega)$ it holds

$$\begin{aligned}
\langle \Delta v, \varphi\rangle_{\mathcal{D}'(\Omega), C_0^\infty(\Omega)} &= \int_\Omega v\,\Delta\varphi \,\mathrm{d}x = \sum_{i=1}^n \int_\Omega \left(\int_\Omega \frac{\partial}{\partial y_i}\phi(x,y)\, f_i(y)\,\mathrm{d}y\right) \Delta\varphi(x)\,\mathrm{d}x \\
&= \sum_{i=1}^n \int_\Omega \left(\int_\Omega \frac{\partial}{\partial y_i}\phi(x,y)\,\Delta\varphi(x)\,\mathrm{d}x\right) f_i(y)\,\mathrm{d}y \\
&= \sum_{i=1}^n \int_\Omega \left(\int_\Omega \frac{\partial}{\partial y_i}(\Delta\phi_y(x))\,\varphi(x)\,\mathrm{d}x\right) f_i(y)\,\mathrm{d}y = 0,
\end{aligned}$$

meaning $\Delta v = 0$ in $\mathcal{D}'(\Omega)$. So $v$ is harmonic and consequently $v \in C^\infty(\Omega)$. Observe that

$$\partial_j v(x) = \sum_{i=1}^n \int_\Omega \frac{\partial^2\phi(x,y)}{\partial x_j \partial y_i}\, f_i(y)\,\mathrm{d}y$$

for $x \in \Omega$ and $j \in \{1,\dots,n\}$. If $V \subset\subset \Omega$ is given, according to Section 5.4 there is a constant $c$ depending on $V$ such that $|\frac{\partial^2}{\partial x_j\partial y_i}\phi(x,y)| \leq c$ for all $(x,y) \in V \times \Omega$ and all $i,j \in \{1,\dots,n\}$. Hence we conclude for $x \in V$ that

$$|\nabla v(x)| \leq c\,\|f\|_{L^1(\Omega;\mathbb{R}^n)},$$

where $c = c(n,V)$. Finally one obtains

$$\|\nabla v\|_{L^\infty(V;\mathbb{R}^n)} \leq c\,\|f\|_{L^1(\Omega;\mathbb{R}^n)}.$$

This finishes the proof. □

Now we have gathered all the means to prove Proposition 6.10.

*PROOF of Proposition 6.10.* From Lemma 5.18 we conclude that $u$ is represented by the following explicit formula

$$u(x) = \int_\Omega \widetilde{G}(x,y) \cdot f(y)\,\mathrm{d}y = w(x) - v(x), \qquad x \in \Omega.$$

Here $w$ and $v$ are as defined in Lemma 6.11 and Lemma 6.12, respectively, if $f$ is assumed to be trivially extended to $\mathbb{R}^n$ by zero. Let $V \subset\subset \mathbb{R}^n$ be given. Then, by (6.21) and (6.28) there is a constant $c = c(n, V)$ such that the following estimate holds,

$$\begin{aligned} \|\nabla u\|_{L^1_w(V;\mathbb{R}^n)} &\leq c\left(\|\nabla v\|_{L^\infty(V;\mathbb{R}^n)} + \|\nabla w\|_{L^1_w(\mathbb{R}^n;\mathbb{R}^n)}\right) \\ &\leq c\left(\|f\|_{L^1(\Omega;\mathbb{R}^n)} + \|f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}\right) \leq c\,\|f\|_{L^1(\Omega;\mathbb{R}^n)}. \end{aligned}$$

This is exactly the claim. □

## 6.4. Equi-integrability

In this section we are going to derive results on equi-integrable functions which are essentially needed for the proof of the compensated compactness in Theorem 6.20. While the classical notion of equi-integrability is well-known, we will be working with its obvious generalization to the $L^p$-setting.

**Definition 6.13 (Equi-integrability in $L^p$)** *Let $\Omega \subset \mathbb{R}^n$ be open and $1 \leq p < \infty$. A sequence $\{f^k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ is called equi-integrable in $L^p$, if for every $\varepsilon > 0$ there exists a $\lambda = \lambda(\varepsilon) > 0$ such that*

$$\int_E |f^k(x)|^p\,\mathrm{d}x < \varepsilon$$

*for all $k \in \mathbb{N}$ and all measurable sets $E \subset \Omega$ with $|E| < \lambda$.*

Proving the following lemma provides the insight that equi-integrability considerably improves convergence properties of sequences.

**Lemma 6.14** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $1 \leq p < \infty$. If the sequence $\{f^k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ is equi-integrable in $L^p$ and $f^k \to 0$ in $L^1_w(\Omega;\mathbb{R}^m)$ as $k \to \infty$, then $f^k \to 0$ in $L^p(\Omega;\mathbb{R}^m)$ for $k \to \infty$.*

*PROOF.* Let $\varepsilon > 0$. By the $L^p$-equi-integrability of $\{f^k\}_{k\in\mathbb{N}}$ there is a $\lambda = \lambda(\varepsilon)$ such that

$$\int_E |f^k|^p\,\mathrm{d}x < \frac{\varepsilon}{2} \tag{6.29}$$

for all measurable $E \subset \Omega$ with $|E| < \lambda$ and all $k \in \mathbb{N}$. Due to the $L^1_w$-convergence of $\{f^k\}_{k\in\mathbb{N}}$ there is $N = N(\varepsilon) \in \mathbb{N}$ such that for all $k \geq N$

$$\left|\left\{x \in \Omega \, : \, |f^k(x)|^p > \frac{\varepsilon}{2|\Omega|}\right\}\right| \leq \lambda. \tag{6.30}$$

This inequality is an immediate consequence of Remark 5.20, 5. applied with $\delta = \left(\lambda\frac{2|\Omega|}{\varepsilon}\right)^{1/p}$ and $t = \left(\frac{\varepsilon}{2|\Omega|}\right)^{1/p}$. Joining (6.29) and (6.30) together finally gives

$$\begin{aligned}
\|f^k\|^p_{L^p(\Omega;\mathbb{R}^m)} &= \int_\Omega |f^k|^p \,\mathrm{d}x = \int_{\{|f^k|^p > \frac{\varepsilon}{2|\Omega|}\}} |f^k|^p \,\mathrm{d}x + \int_{\{|f^k|^p \leq \frac{\varepsilon}{2|\Omega|}\}} |f^k|^p \,\mathrm{d}x \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2|\Omega|}|\Omega| = \varepsilon
\end{aligned}$$

for all $k \geq N$, which proves the claim. □

The next proposition says that uniformly bounded sequences in $L^p$ can be approximated by $L^p$-equi-integrable ones with respect to strong $L^q$-convergence provided $q < p$. This is basically a consequence of the biting lemma, which we quote here from [70, Theorem 3.3].

**Lemma 6.15 (Biting lemma)** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose $\{f^j\}_{j\in\mathbb{N}}$ is a bounded sequence in $L^1(\Omega;\mathbb{R}^m)$. Then there are a subsequence $\{f^{j_k}\}_{k\in\mathbb{N}}$ and a sequence $\{B^k\}_{k\in\mathbb{N}}$ of subsets of $\Omega$ decreasing to $\emptyset$ such that the sequence $\{\chi_{(\Omega\setminus B^k)} f^{j_k}\}_{k\in\mathbb{N}}$ is equi-integrable in $L^1$.*

**Proposition 6.16** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $1 < p < \infty$. If $\{f^j\}_{j\in\mathbb{N}}$ is uniformly bounded in $L^p(\Omega;\mathbb{R}^m)$, then there are a subsequence $\{f^{j_k}\}_{k\in\mathbb{N}}$ and a $L^p$-equi-integrable sequence $\{g^k\}_{k\in\mathbb{N}}$ such that for all $q \in [1, p)$,*

$$f^{j_k} - g^k \to 0 \qquad \text{in } L^q(\Omega;\mathbb{R}^m) \qquad \text{as } k \to \infty.$$

*PROOF.* For $j \in \mathbb{N}$ we set $h^j = (f^j)^p$ on $\Omega$. By assumption $\{h^j\}_{j\in\mathbb{N}}$ is uniformly bounded in $L^1(\Omega;\mathbb{R}^m)$. According to the biting lemma, we find a subsequence $\{h^{j_k}\}_{k\in\mathbb{N}}$ and a sequence $\{B^k\}_{k\in\mathbb{N}}$ of subsets of $\Omega$ with $|B^k| \to 0$ for $k \to \infty$, such that $\{\chi_{(\Omega\setminus B^k)} h^{j_k}\}_{k\in\mathbb{N}}$ is equi-integrable in $L^1$. With the help of the sets $B^k$ we can decompose $f^{j_k}$ in the following way,

$$f^{j_k} = \chi_{(\Omega\setminus B^k)} f^{j_k} + \chi_{B^k} f^{j_k} =: g^k + b^k.$$

The next calculation reveals that $b^k \to 0$ in $L^q(\Omega;\mathbb{R}^m)$ as $k \to \infty$ for every $1 \leq q < p$. Indeed, by Hölder's inequality

$$\|b^k\|^q_{L^q(\Omega;\mathbb{R}^m)} = \int_\Omega |b^k|^q \,\mathrm{d}x = \int_{B^k} |f^{j_k}|^q \,\mathrm{d}x \leq \|f^{j_k}\|^q_{L^p(\Omega;\mathbb{R}^m)} |B^k|^{\frac{p-q}{p}} \leq c\,|B^k|^{\frac{p-q}{p}}$$

and $|B^k| \to 0$ by assumption. Besides we infer from $(g^k)^p = \chi_{(\Omega \setminus B^k)}(f^{j_k})^p = \chi_{(\Omega\setminus B^k)} h^{j_k}$ that $\{g^k\}_{k\in\mathbb{N}}$ is equi-integrable in $L^p$. Finally the claim follows, because $b^k = f^{j_k} - g^k$ for $k \in \mathbb{N}$. □

Next we want to focus on a sequence of Dirichlet problems with the divergence of $L^p$-functions as right-hand sides. The main result will be that $L^p$-equi-integrability of the right-hand side functions carries over to the gradients of the corresponding solutions, so that they are equi-integrable as well.

**Proposition 6.17** *Suppose $\Omega \subset \mathbb{R}^n$ is open and bounded with boundary $\partial\Omega$ of class $C^1$ and $\{f^k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^n)$ is equi-integrable in $L^p$ with $1 < p < \infty$. Let $u^k \in W_0^{1,p}(\Omega)$ denote the weak solution of*

$$\begin{aligned} \Delta u^k &= \operatorname{div} f^k && \text{in } \Omega, \\ u^k &= 0 && \text{on } \partial\Omega \end{aligned}$$

*for $k \in \mathbb{N}$. Then, the sequence of gradients $\{\nabla u^k\}_{k\in\mathbb{N}}$ is equi-integrable in $L^p$.*

*PROOF.* Let $\varepsilon > 0$ be given and take $\mu > 0$ to be chosen later. Since $\{f^k\}_{k\in\mathbb{N}}$ is equi-integrable in $L^p$, there is a $\sigma > 0$ such that

$$\int_G |f^k|^p \, \mathrm{d}x \le \frac{\mu}{2^p}\varepsilon \tag{6.31}$$

for all $k \in \mathbb{N}$ and all measurable $G \subset \Omega$ with $|G| < \sigma$. Note that the $L^p$-equi-integrability of $\{f^k\}_{k\in\mathbb{N}}$ implies that this sequence is uniformly bounded in $L^p(\Omega;\mathbb{R}^n)$. For $k \in \mathbb{N}$ we define

$$\Omega^k = \Big\{ x \in \Omega \ : \ |f^k(x)| > \frac{1}{\sigma^{1/p}} \sup_{j\in\mathbb{N}} \|f^j\|_{L^p(\Omega;\mathbb{R}^n)} \Big\}.$$

In view of $\sup_{j\in\mathbb{N}} \|f^j\|^p_{L^p(\Omega;\mathbb{R}^n)} |\Omega^k| \le \sigma \int_\Omega |f^k|^p \, \mathrm{d}x$ we have $|\Omega^k| \le \sigma$ for all $k \in \mathbb{N}$ and by (6.31),

$$\int_{\Omega^k} |f^k|^p \, \mathrm{d}x \le \frac{\mu}{2^p}\varepsilon. \tag{6.32}$$

Now we use $\Omega^k$ to decompose $f^k$ into a bounded part and one that can be controlled by the equi-integrability of $\{f^k\}_{k\in\mathbb{N}}$. To this end we set $g^k = (1 - \chi_{\Omega^k}) f^k$ and $h^k = \chi_{\Omega^k} f^k$ for all $k \in \mathbb{N}$, so that $f^k = g^k + h^k$ with $h^k \in L^p(\Omega;\mathbb{R}^n)$ and $g^k \in L^\infty(\Omega;\mathbb{R}^n)$. The latter implies $g^k \in L^q(\Omega;\mathbb{R}^n)$ for all $q \in [1,\infty)$, in particular $g^k \in L^{2p}(\Omega;\mathbb{R}^n)$.
Suppose $v^k \in W_0^{1,2p}(\Omega)$ and $w^k \in W_0^{1,p}(\Omega)$ are the unique solutions of the boundary-value problems

$$\begin{aligned} \Delta v^k &= \operatorname{div} g^k && \text{in } \Omega, \\ v^k &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{6.33}$$

and

$$\begin{aligned} \Delta w^k &= \operatorname{div} h^k && \text{in } \Omega, \\ w^k &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{6.34}$$

respectively. Then, $u^k = v^k + w^k$ for $k \in \mathbb{N}$.
As a next step we apply elliptic $L^p$-theory in terms of Section 6.3 to both (6.33) and (6.34). Precisely, we conclude from Theorem 6.9 that $\|\nabla w^k\|^p_{L^p(\Omega;\mathbb{R}^n)} \leq c\,\|h^k\|^p_{L^p(\Omega;\mathbb{R}^n)}$ with $c = c(n,p,\Omega)$. If we consider some measurable set $E \subset \Omega$, this estimate together with (6.32) leads to

$$\begin{aligned} \int_E |\nabla w^k|^p \,\mathrm{d}x &\leq \int_\Omega |\nabla w^k|^p \,\mathrm{d}x \leq c \int_\Omega |h^k|^p \,\mathrm{d}x \\ &= c \int_{\Omega^k} |f^k|^p \,\mathrm{d}x \leq \frac{c\mu}{2^p}\varepsilon \leq \frac{\varepsilon}{2^p}, \end{aligned} \tag{6.35}$$

for all $k \in \mathbb{N}$, where $\mu > 0$ was chosen small enough to satisfy $c\mu \leq 1$.
Similar arguments of $L^p$-theory applied to (6.33) yield $\nabla v^k \in L^{2p}(\Omega;\mathbb{R}^n)$ with the estimate $\|\nabla v^k\|_{L^{2p}(\Omega;\mathbb{R}^n)} \leq c\,\|g^k\|_{L^{2p}(\Omega;\mathbb{R}^n)}$, where $c = c(n,p,\Omega)$. Using Hölder's inequality finally gives

$$\begin{aligned} \int_E |\nabla v^k|^p \,\mathrm{d}x &\leq |E|^{1/2} \Big( \int_\Omega |\nabla v^k|^{2p} \,\mathrm{d}x \Big)^{1/2} \\ &\leq c\,|E|^{1/2} \Big( \int_\Omega |g^k|^{2p} \,\mathrm{d}x \Big)^{1/2} \leq c_0 \frac{|\Omega|^{1/2}}{\sigma} \sup_{j\in\mathbb{N}} \|f^j\|^p_{L^p(\Omega;\mathbb{R}^n)} |E|^{1/2}. \end{aligned} \tag{6.36}$$

The last inequality is true, since $\{g^k\}_{k\in\mathbb{N}}$ is uniformly bounded by $\frac{1}{\sigma^{1/p}} \sup_{j\in\mathbb{N}} \|f^j\|_{L^p(\Omega;\mathbb{R}^n)}$ in $L^\infty(\Omega)$ by definition.
Let us now choose $\delta > 0$ such that $c_o \frac{|\Omega|^{1/2}}{\sigma} \sup_{j\in\mathbb{N}} \|f^j\|^p_{L^p(\Omega;\mathbb{R}^n)} \delta^{1/2} \leq \frac{\varepsilon}{2^p}$. According to (6.35) and (6.36) it holds for $E \subset \Omega$ measurable with $|E| < \delta$ and all $k \in \mathbb{N}$ that

$$\begin{aligned} \int_E |\nabla u^k|^p \,\mathrm{d}x &= \int_E |\nabla v^k + \nabla w^k|^p \,\mathrm{d}x \\ &\leq 2^{p-1} \int_E |\nabla v^k|^p \,\mathrm{d}x + 2^{p-1} \int_E |\nabla w^k|^p \,\mathrm{d}x \\ &< 2^{p-1} c_0 \frac{|\Omega|^{1/2}}{\sigma} \sup_{j\in\mathbb{N}} \|f^j\|^p_{L^p(\Omega;\mathbb{R}^n)}\, \delta^{1/2} + \frac{\varepsilon}{2} \leq \varepsilon, \end{aligned}$$

which is exactly what we had to prove. □

## 6.5. Compensated compactness

The goal we have in mind is to guarantee that $\det \nabla u^k$ converges to $\det \nabla u$ as $k$ tends to infinity, provided $\{u^k\}_{k\in\mathbb{N}}$ is a bounded energy sequence of $E_{\varepsilon_k}$ as in the statement of Theorem 1.2 converging to $u$. This is what we actually mean by the recovery of the incompressibility constraint in the limit. Mathematically speaking, this is a matter of weak continuity. So far problems of this type have been treated effectively with the help of compensated compactness, a concept which was introduced by Murat and Tartar [65, 66, 74, 75] in the late seventies of the last century and which builds upon the div-curl lemma. The latter is formulated in its classical version right below.

**Theorem 6.18 (Div-curl lemma)** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Further let $\{u^k\}_{k\in\mathbb{N}} \subset L^2(\Omega;\mathbb{R}^n)$ and $\{v^k\}_{k\in\mathbb{N}} \subset L^2(\Omega;\mathbb{R}^n)$ be sequences such that $u^k \rightharpoonup u$ in $L^2(\Omega;\mathbb{R}^n)$ and $v^k \rightharpoonup v$ in $L^2(\Omega;\mathbb{R}^n)$ for $k \to \infty$ and suppose that*

$$\{\operatorname{div} u^k\}_{k\in\mathbb{N}} \quad \textit{is compact in } W^{-1,2}(\Omega) \tag{6.37}$$

*and*

$$\{\operatorname{curl} v^k\}_{k\in\mathbb{N}} \quad \textit{is compact in } W^{-1,2}(\Omega;\mathbb{R}^{n\times n}). \tag{6.38}$$

*Then, $u^k \cdot v^k \to u \cdot v$ in $\mathcal{D}'(\Omega)$ as $k \to \infty$.*

By now there have been various generalizations of this result. One of the first was for weakly converging sequences $\{u^k\}_{k\in\mathbb{N}}$ and $\{v^k\}_{k\in\mathbb{N}}$ in $L^p(\Omega;\mathbb{R}^n)$ and $L^q(\Omega;\mathbb{R}^n)$, respectively, where $p, q \in (1,\infty)$ are dual exponents i.e. $\frac{1}{p} + \frac{1}{q} = 1$, with $\{\operatorname{div} u^k\}_{k\in\mathbb{N}}$ compact in $W^{-1,p}(\Omega)$ and $\{\operatorname{curl} v^k\}_{k\in\mathbb{N}}$ compact in $W^{-1,q}(\Omega;\mathbb{R}^{n\times n})$, see for instance [66].
If we go back again to considering a bounded energy sequence $\{u^k\}_{k\in\mathbb{N}}$ of $E_{\varepsilon_k}$ in the 2D setting, we will find in view of Lemma 7.2 that $\nabla u^k$ converges only weakly in $L^1$ due to the non-standard growth behavior of $W_\varepsilon$. However, from the anisotropic nature of $W_\varepsilon$ we can derive a special structure of the bounded energy sequence resulting in the decomposition $\nabla u^k = A^k + H^k$, where $A^k \rightharpoonup \nabla u$ in $L^2$ and $H^k \to 0$ in $L^1$. Now if we try to apply Theorem 6.18 to the sequences formed by the first row and the rotated second row of the 'good' contribution $A^k$, so that their scalar product is exactly $\det A^k$, we observe the following. By the mere $L^1$-convergence of $H^k$ the conditions (6.37) and (6.38) are too strong for the classical div-curl lemma to apply in this context, so that we cannot carry out the limit process in this way.
With this motivation in mind we prove a generalization of Theorem 6.18 adequate to overcome the aforementioned difficulty. In fact, a div-curl lemma for sequences whose divergence and curl are compact as functionals on Lipschitz functions is needed.

### 6.5.1. A generalized div-curl lemma

The result we are going to prove now acts on the weaker assumption that both $\{\operatorname{div} u^k\}_{k\in\mathbb{N}}$ and $\{\operatorname{curl} v^k\}_{k\in\mathbb{N}}$ are, roughly speaking, only compact in $W^{-1,1}$. By $W^{-1,1}(\Omega)$ we denote here the dual space of $W_0^{1,\infty}(\Omega)$. Additionally, we postulate that $\{u^k \cdot v^k\}_{k\in\mathbb{N}}$ is equi-integrable.
To be more precise about the exact requirements let us define the subsequent stronger notion of convergence for sequences in $W^{-1,1}$.

**Definition 6.19** **($\Box$-convergence in $W^{-1,1}$)** *Let $\Omega \subset \mathbb{R}^n$ be an open set. A sequence $\{f^k\}_{k\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$ is said to $\Box$-converge to $0$ in $W^{-1,1}(\Omega)$, in symbols*

$$f^k \xrightarrow{\Box} 0 \qquad \text{in } \; W^{-1,1}(\Omega) \qquad \text{as } k \to \infty,$$

*if there exists a sequence of functions $\{g^k\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ with $f^k = \operatorname{div} g^k$ in $\mathcal{D}'(\Omega)$ for all $k \in \mathbb{N}$ and $g^k \to 0$ in $L^1(\Omega)$ for $k \to \infty$.*
*For vector-valued distributions $\{f^k\}_{k\in\mathbb{N}} \subset \mathcal{D}'(\Omega;\mathbb{R}^m)$ we say that the sequence tends to $0$ in $W^{-1,1}(\Omega;\mathbb{R}^m)$ in the sense of $\Box$-convergence, if each component converges to $0$ in the foregoing sense.*

With this definition the generalized div-curl lemma can be stated as follows.

**Theorem 6.20** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and $p, q \in (1,\infty)$ with $1/p + 1/q = 1$. Suppose $\{u^k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^n)$ and $\{v^k\}_{k\in\mathbb{N}} \subset L^q(\Omega;\mathbb{R}^n)$ are sequences such that $u^k \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^n)$ and $v^k \rightharpoonup v$ in $L^q(\Omega;\mathbb{R}^n)$ for $k \to \infty$ with*

$$\operatorname{div} u^k \xrightarrow{\Box} 0 \quad \text{in } W^{-1,1}(\Omega) \qquad \text{and} \qquad \operatorname{curl} v^k \xrightarrow{\Box} 0 \quad \text{in } W^{-1,1}(\Omega;\mathbb{R}^{n\times n}). \tag{6.39}$$

*Finally assume that $\{u^k \cdot v^k\}_{k\in\mathbb{N}}$ is equi-integrable. Then,*

$$u^k \cdot v^k \rightharpoonup u \cdot v \qquad \text{in } L^1(\Omega) \qquad \text{for } k \to \infty. \tag{6.40}$$

**Remark 6.21** *1. The idea for this theorem was originally brought up by Stefan Müller, who also inspired the outline of the proof via Helmholtz decomposition and $L^p$-theory given below. In fact, Conti, Dolzmann and Müller recently managed to enhance this theorem in the sense that they only need to postulate common $W^{-1,1}$-convergence for the sequences $\{\operatorname{div} u^k\}_{k\in\mathbb{N}}$ and $\{\operatorname{curl} v^k\}_{k\in\mathbb{N}}$, which seems more natural. In [22] they give an elegant proof based on a different approach using Lipschitz-truncations of Sobolev functions along the lines of [35].*
*2. Notice that the equi-integrability of $\{u^k \cdot v^k\}_{k\in\mathbb{N}}$ is a necessary condition. For the one-dimensional counterexample of a Fakir's carpet we refer to [22, Remarks].*

### 6.5.2. Proofs

The key ingredient for the proof of Theorem 6.20 is the observation that the equi-integrability of $\{u^k \cdot v^k\}_{k\in\mathbb{N}}$ allows us to construct modified sequences $\{\tilde{u}^k\}_{k\in\mathbb{N}}$ and $\{\tilde{v}^k\}_{k\in\mathbb{N}}$ which are $L^p$- and $L^q$-equi-integrable, respectively, so that $\tilde{u}^k \cdot \tilde{v}^k$ still has $u \cdot v$ as a weak limit. This construction relies on the biting lemma, Lemma 6.15, and is essential to reduce the problem to a setting where the classical div-curl lemma applies.
Before we focus directly on the verification of Theorem 6.20, let us give two lemmata illustrating how the rather weak condition of (6.39) improves considerably in view of the equi-integrability of $\{u^k\}_{k\in\mathbb{N}}$ and $\{v^k\}_{k\in\mathbb{N}}$.

**Lemma 6.22** *Let $1 < p < \infty$, $x \in \mathbb{R}^n$ and $\rho > 0$. Suppose $\{u^k\}_{k\in\mathbb{N}} \subset L^p(B(x,\rho);\mathbb{R}^n)$ is an equi-integrable sequence in $L^p$ with $\operatorname{div} u^k \xrightarrow{\square} 0$ in $W^{-1,1}(B(x,\rho))$ as $k \to \infty$. Then for every $V \subset\subset B(x,\rho)$ there exists a sequence $\{a^k\}_{k\in\mathbb{N}} \subset L^p(V;\mathbb{R}^n)$ such that*

$$\operatorname{div} a^k = 0 \quad \textit{in } W^{-1,p}(V) \quad \textit{for all } k \in \mathbb{N}$$

*and*

$$a^k - u^k \to 0 \qquad \textit{in } L^p(V;\mathbb{R}^n) \qquad \textit{for } k \to \infty.$$

*PROOF.* By scaling and translation we may assume that $B(x,\rho) = B(0,1)$. Let $V \subset\subset B(0,1)$ and set $r = 1 - \frac{1}{2}\operatorname{dist}(V, \partial B(0,1))$. Then $V \subset\subset B(0,r) \subset\subset B(0,1)$.
According to Theorem 6.9 let $w^k \in W_0^{1,p}(B(0,r))$ be the unique solution of

$$\begin{aligned} \Delta w^k &= \operatorname{div} u^k && \text{in } B(0,r), \\ w^k &= 0 && \text{on } \partial B(0,r) \end{aligned} \tag{6.41}$$

for $k \in \mathbb{N}$. Defining $a^k = u^k - \nabla w^k \in L^p(B(0,r);\mathbb{R}^n)$ provides for all $k \in \mathbb{N}$ the following decomposition of $u^k$ on $B(0,r)$,

$$u^k = \nabla w^k + a^k \quad \text{in } L^p(B(0,r);\mathbb{R}^n) \quad \text{with } \operatorname{div} a^k = 0 \text{ in } W^{-1,p}(B(0,r)). \tag{6.42}$$

An additive decomposition of a function into a gradient term and a divergence-free term, like (6.42), is usually referred to as Helmholtz-decomposition.
In view of (6.41) and the $L^p$-equi-integrability of $\{u^k\}_{k\in\mathbb{N}}$ we infer from Proposition 6.17 that

$$\{\nabla w^k\}_{k\in\mathbb{N}} \subset L^p(B(0,r);\mathbb{R}^n) \text{ is equi-integrable in } L^p. \tag{6.43}$$

Moreover, by Definition 6.19 there exists a sequence $\{g^k\}_{k\in\mathbb{N}} \subset L^1(B(0,1);\mathbb{R}^n)$ with $g^k \to 0$ in $L^1(B(0,1);\mathbb{R}^n)$ and

$$\operatorname{div} g^k = \operatorname{div} u^k \quad \text{in } \mathcal{D}'(B(0,1)) \quad \text{for all } k \in \mathbb{N}. \tag{6.44}$$

Hence, $w^k \in W_0^{1,p}(B(0,r))$ as defined through (6.41) is a weak solution of the Dirichlet problem

$$\begin{aligned} \Delta w^k &= \operatorname{div} g^k \quad \text{in } B(0,r), \\ w^k &= 0 \qquad\quad \text{on } \partial B(0,r), \end{aligned} \tag{6.45}$$

as well. Since the right-hand side of (6.45) is lacking the required smoothness to apply the weak-$L^1$ estimate of Proposition 6.10, we make use of convolution techniques to obtain smooth approximating functions for $g^k$. Note that in the following the same notation is used for functions defined on $B(0,r)$ and their trivial extensions to $\mathbb{R}^n$ by zero. For $l \in \mathbb{N}$ let $\eta_l : \mathbb{R}^n \to \mathbb{R}$ be given by $\eta_l(y) = l^n \eta(ly)$ with $\eta$ the standard convolution kernel, so that $\eta_l \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \eta_l \subset B(0,1/l)$ and $\int_{\mathbb{R}^n} \eta_l \,\mathrm{d}y = 1$.
We define the functions $g_l^k = (g^k * \eta_l)|_{B(0,r)} \in C^\infty(\overline{B(0,r)};\mathbb{R}^n)$ and $u_l^k = (u^k * \eta_l)|_{B(0,r)} \in C^\infty(\overline{B(0,r)};\mathbb{R}^n)$ for $l,k \in \mathbb{N}$. The subsequent arguments will show that for all $k \in \mathbb{N}$ and $l \in \mathbb{N}$ sufficiently large depending on $r$,

$$\operatorname{div} g_l^k = \operatorname{div} u_l^k \quad \text{in } \mathcal{D}'(B(0,r)), \tag{6.46}$$

or more exactly $\langle \operatorname{div} g_l^k, \varphi\rangle = \langle \operatorname{div} u_l^k, \varphi\rangle$ for all $\varphi \in C_0^\infty(B(0,r))$. To see this we infer from Fubini-Tonelli's Theorem and the properties of convolution that for $l \in \mathbb{N}$ large enough to satisfy $r + 1/l < 1$,

$$\begin{aligned} \langle \operatorname{div} g_l^k, \varphi\rangle &= -\int_{B(0,r)} g_l^k \cdot \nabla\varphi \,\mathrm{d}y = -\int_{B(0,r)} (g^k * \eta_l) \cdot \nabla\varphi \,\mathrm{d}y \\ &= -\int_{B(0,1)} g^k \cdot (\nabla\varphi * \eta_l) \,\mathrm{d}y = -\int_{B(0,1)} g^k \cdot \nabla(\varphi * \eta_l) \,\mathrm{d}y. \end{aligned} \tag{6.47}$$

Since $(\varphi * \eta_l) \in C_0^\infty(B(0,1))$ for the above choice of $l$, we learn from (6.44) that $\int_{B(0,1)} g^k \cdot \nabla(\varphi * \eta_l) \,\mathrm{d}y = \int_{B(0,1)} u^k \cdot \nabla(\varphi * \eta_l) \,\mathrm{d}y$. Applying the same line of reasoning as in (6.47) backwards with $g^k$ and $g_l^k$ replaced by $u^k$ and $u_l^k$ yields the postulated equality (6.46).
For $k,l \in \mathbb{N}$ let $w_l^k$ denote the unique classical solution of the boundary value problem

$$\begin{aligned} \Delta w_l^k &= \operatorname{div} u_l^k \quad \text{in } B(0,r), \\ w_l^k &= 0 \qquad\quad \text{on } \partial B(0,r). \end{aligned} \tag{6.48}$$

By (6.46) the function $w_l^k$ also solves

$$\begin{aligned} \Delta w_l^k &= \operatorname{div} g_l^k \quad \text{in } B(0,r), \\ w_l^k &= 0 \qquad\quad \text{on } \partial B(0,r) \end{aligned} \tag{6.49}$$

for all $k \in \mathbb{N}$ provided $l \in \mathbb{N}$ is large enough. This implies that $w_l^k \in C^\infty(\overline{B(0,r)})$ is a classical solution of (6.49). Then, the application of Proposition 6.10 to (6.49) provides

$$\|\nabla w_l^k\|_{L^1_w(V;\mathbb{R}^n)} \le c\,\|g_l^k\|_{L^1(B(0,r);\mathbb{R}^n)} \le c\,\|g^k\|_{L^1(B(0,1);\mathbb{R}^n)} \tag{6.50}$$

with a constant $c = c(n, V)$ for all $k \in \mathbb{N}$ and for sufficiently large $l$. Further, subtracting (6.41) from (6.48) gives

$$\begin{aligned} \Delta(w_l^k - w^k) &= \operatorname{div}(u_l^k - u^k) && \text{in } B(0,r), \\ w_l^k - w^k &= 0 && \text{on } \partial B(0,r), \end{aligned}$$

so that Theorem 6.9 implies

$$\|\nabla w_l^k - \nabla w^k\|_{L^p(B(0,r);\mathbb{R}^n)} \le c\,\|u_l^k - u^k\|_{L^p(B(0,r);\mathbb{R}^n)} \tag{6.51}$$

with $c = c(n, p, r)$ for all $k, l \in \mathbb{N}$.
Finally, joining (6.50) and (6.51) together leads to

$$\begin{aligned} \|\nabla w^k\|_{L^1_w(V;\mathbb{R}^n)} &\le c\left(\|\nabla w_l^k\|_{L^1_w(V;\mathbb{R}^n)} + \|\nabla w_l^k - \nabla w^k\|_{L^1_w(V;\mathbb{R}^n)}\right) \\ &\le c\left(\|g^k\|_{L^1(B(0,1);\mathbb{R}^n)} + \|u_l^k - u^k\|_{L^p(B(0,r);\mathbb{R}^n)}\right) \end{aligned}$$

provided $l$ is large enough. Hence, by the approximative properties of convolution, see i.e. [31, Appendix C.4], taking the limit $l \to \infty$ results in

$$\|\nabla w^k\|_{L^1_w(V;\mathbb{R}^n)} \le c\,\|g^k\|_{L^1(B(0,1);\mathbb{R}^n)} \tag{6.52}$$

for all $k \in \mathbb{N}$, where the constant $c > 0$ depends on $n$, $p$ and $V$.
In view of (6.43), (6.52) and the fact that $g^k \to 0$ in $L^1(B(0,1);\mathbb{R}^n)$ one can infer from Lemma 6.14 that

$$\nabla w^k \to 0 \quad \text{in} \quad L^p(V;\mathbb{R}^n) \quad \text{as } k \to \infty.$$

The claim follows by considering the Helmholtz decomposition (6.42). $\square$

Next we present the analogous result for equi-integrable sequences with compact curl.

**Lemma 6.23** *Let $1 < q < \infty$, $x \in \mathbb{R}^n$ and $\rho > 0$. Suppose $\{v^k\}_{k\in\mathbb{N}} \subset L^q(B(x,\rho);\mathbb{R}^n)$ is an equi-integrable sequence in $L^q$ with $\operatorname{curl} v^k \xrightarrow{\square} 0$ in $W^{-1,1}(B(x,\rho);\mathbb{R}^{n\times n})$. Then for every $V \subset\subset B(x,\rho)$ there exists a sequence $\{b^k\}_{k\in\mathbb{N}} \subset L^q(V;\mathbb{R}^n)$ such that*

$$\operatorname{curl} b^k \to 0 \quad \text{in } W^{-1,q}(V;\mathbb{R}^{n\times n}), \tag{6.53}$$

*and*

$$b^k - v^k \to 0 \quad \text{in } L^q(V;\mathbb{R}^n) \quad \text{as } k \to \infty. \tag{6.54}$$

*PROOF.* By scaling and translation we assume without loss of generality that $B(x,\rho) = B(0,1)$. Let $V \subset\subset B(0,1)$ and set $r = 1 - \frac{1}{2}\operatorname{dist}(V, \partial B(0,1))$. Then $V \subset\subset B(0,r) \subset\subset B(0,1)$.

First we will show the result for $n = 2$. By this we want to illustrate how the problem can be traced back to a situation for which the statement of Lemma 6.22 is directly applicable. After that the general case will be treated, which is more technical but can be reduced to the setting of the previous lemma as well.
Here the crucial observation to simplify the argumentation is that $\operatorname{curl} v^k$ can be expressed with the help of the divergence of the rotated $v^k$ in the form

$$\operatorname{curl} v^k = -\operatorname{div}(Jv^k). \tag{6.55}$$

Recall that $J$ stands for the counterclockwise rotation by $\pi/2$ in the plane and note that we identify $\operatorname{curl} v^k \in \mathbb{R}^{2\times 2}$ with $(\operatorname{curl} v^k)_{12} \in \mathbb{R}$ for simplicity.
With $z^k \in W_0^{1,q}(B(0,r))$ being the unique weak solution of

$$\begin{aligned} \Delta z^k &= \operatorname{div}(Jv^k) && \text{in } B(0,r), \\ z^k &= 0 && \text{on } \partial B(0,r) \end{aligned} \tag{6.56}$$

for every $k \in \mathbb{N}$ and by defining $\tilde{b}^k = Jv^k - \nabla z^k \in L^q(B(0,r);\mathbb{R}^n)$ we again end up with a decomposition of Helmholtz type,

$$Jv^k = \nabla z^k + \tilde{b}^k \quad \text{in } L^q(B(0,r);\mathbb{R}^n) \qquad \text{with} \quad \operatorname{div} \tilde{b}^k = 0 \quad \text{in } \mathcal{D}'(B(0,r)).$$

Regarding $\operatorname{div}(Jv^k) = -\operatorname{curl} v^k \xrightarrow{\square} 0$ in $W^{-1,1}(B(0,1))$ the sequence $\{Jv_k\}_{k\in\mathbb{N}}$ meets all the requirements of Lemma 6.22. So we have $Jv^k - \tilde{b}^k \to 0$ in $L^q(V;\mathbb{R}^n)$, which is equivalent to $v^k - J\tilde{b}^k \to 0$ in $L^q(V;\mathbb{R}^n)$ for $k \to \infty$. Finally, setting $b^k = J\tilde{b}^k$ results in

$$v^k - b^k \to 0 \text{ in } \; L^q(V;\mathbb{R}^n) \quad \text{and} \quad \operatorname{curl} b^k = \operatorname{div} \tilde{b}^k = 0 \text{ in } \; W^{-1,q}(B(0,r)). \tag{6.57}$$

This proves the claim in two dimensions.
In higher dimensions the relation between div and curl is more complicated than in 2D, so that we introduce the following generalization of $J$. For $i, j \in \{1, \dots, n\}$ let us define $J_{ij} \in \mathbb{R}^{n\times n}$ through

$$(J_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha} \quad \text{for } \alpha, \beta \in \{1, \dots, n\}.$$

Then,

$$(\operatorname{curl} v^k)_{ij} = \operatorname{div}(J_{ij} v^k) \tag{6.58}$$

for all $i, j \in \{1, \dots, n\}$ and $k \in \mathbb{N}$. If $z_{ij}^k \in W_0^{1,q}(B(0,r))$ denotes the unique weak solution of

$$\begin{aligned} \Delta z_{ij}^k &= \operatorname{div}(J_{ij} v^k) && \text{in } B(0,r), \\ z_{ij}^k &= 0 && \text{on } \partial B(0,r), \end{aligned} \tag{6.59}$$

we have

$$J_{ij}v^k = \nabla z_{ij}^k + \tilde{b}_{ij}^k \qquad \text{with} \qquad \operatorname{div} \tilde{b}_{ij}^k = 0 \;\; \text{in} \;\; W^{-1,q}(B(0,r))$$

for $i, j \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$. Due to $\operatorname{div}(J_{ij}v^k) = (\operatorname{curl} v^k)_{ij} \xrightarrow{\square} 0$ in $W^{-1,1}(B(0,1))$ we obtain $\nabla z_{ij}^k \to 0$ in $L^q(V; \mathbb{R}^n)$ as in the proof of Lemma 6.22. Besides, for all $m \in \{1, \ldots, n\}$ with $m \neq i$ and $m \neq j$ it holds $(J_{ij}v^k)_m = 0$, which provides the additional information that

$$(\tilde{b}_{ij}^k)_m \to 0 \quad \text{in } L^q(V) \quad \text{as } k \to \infty. \tag{6.60}$$

We now rewrite $v^k$ as $v^k = -\frac{1}{2(n-1)} \sum_{i,j=1}^n J_{ij}J_{ij}v^k$ and define $\tilde{b}^k = -\frac{1}{2(n-1)} \sum_{i,j=1}^n J_{ij}\tilde{b}_{ij}^k$. Then,

$$\tilde{b}^k - v^k = \frac{1}{2(n-1)} \sum_{i,j=1}^n J_{ij} \nabla z_{ij}^k \to 0 \quad \text{in } L^q(V; \mathbb{R}^n) \quad \text{as } k \to \infty. \tag{6.61}$$

After straight forward calculation we find for $m, l \in \{1, \ldots, n\}$,

$$(\operatorname{curl} \tilde{b}^k)_{ml} = \frac{1}{n-1} \left( w_{ml}^k + (n-2)(\operatorname{curl} v^k)_{ml} \right),$$

where $w_{ml}^k = -\sum_{j=1, j\neq m, j\neq l}^n \partial_j \left[ (\tilde{b}_{ml}^k)_j + (\tilde{b}_{lj}^k)_m + (\tilde{b}_{jm}^k)_l \right]$. Note that as a consequence of (6.60),

$$w_{ml}^k \to 0 \;\; \text{in} \;\; W^{-1,q}(V) \tag{6.62}$$

as $k \to \infty$. Next, we set $b^k = (n-1)\tilde{b}^k + (2-n)\, v^k$ for all $k \in \mathbb{N}$. Considering (6.61) this immediately implies (6.54). Moreover, we get $(\operatorname{curl} b^k)_{ml} = w_{ml}^k$ for all $m, l \in \{1, \ldots, n\}$, so that (6.53) follows from (6.62). $\square$

*PROOF of Theorem 6.20.* For reasons of clarity the proof is divided into four steps.

*Step 1: Reduction of the problem to the unit ball.*
Without loss of generality one can restrict to the case $\Omega = B(0,1)$. To see this let $\Omega \subset \mathbb{R}^n$ be a bounded and open Lipschitz set and assume that the assertion of Theorem 6.20 holds on the unit ball. Then, by scaling and translation the latter is also true on ball-shaped domains of the form $B(x, \rho)$ with $x \in \mathbb{R}^n$ and $\rho > 0$. For $\varphi \in C_0^\infty(\Omega)$ there is a finite number of balls in $\Omega$ that cover $\operatorname{supp} \varphi$, more precisely, we have $N \in \mathbb{N}$ and $B(x_j, \rho_j) \subset \Omega$ for $j \in \{1, \ldots, N\}$ such that $\operatorname{supp} \varphi \subset \bigcup_{j=1}^N B(x_j, \rho_j)$. Take $\{\zeta^j\}_{j\in\{1,\ldots,N\}}$

to be a partition of unity for the sets $\{B(x_j,\rho_j)\}_{j\in\{1,\dots,N\}}$. Consequently, we find

$$\begin{aligned}\int_\Omega (u^k\cdot v^k)\varphi \,\mathrm{d}y &= \sum_{j=1}^N \int_{B(x_j,\rho_j)} (u^k\cdot v^k)\zeta^j\varphi\,\mathrm{d}y\\ &\to \sum_{j=1}^N \int_{B(x_j,\rho_j)} (u\cdot v)\zeta^j\varphi\,\mathrm{d}y = \int_\Omega (u\cdot v)\varphi\,\mathrm{d}y.\end{aligned}$$

Since $\varphi\in C_0^\infty(\Omega)$ was arbitrary, this means $u^k\cdot v^k\to u\cdot v$ in $\mathcal{D}'(\Omega)$. With the sequence of inner products $\{u^k\cdot v^k\}_{k\in\mathbb{N}}$ being equi-integrable and uniformly bounded in $L^1(\Omega)$, it holds (possibly after passing to a subsequence) that $u^k\cdot v^k\rightharpoonup u\cdot v$ in $L^1(\Omega)$. As the limit is independent of the chosen subsequence, the entire sequence converges as well. This is exactly the statement of Theorem 6.20.

*Step 2: Reduction to $L^p$- and $L^q$-equi-integrable sequences $\{\tilde u^k\}_{k\in\mathbb{N}}$ and $\{\tilde v^k\}_{k\in\mathbb{N}}$.*
By assumption the sequences $\{|u^k|^p\}_{k\in\mathbb{N}}$ and $\{|v^k|^q\}_{k\in\mathbb{N}}$ are bounded in $L^1(B(0,1))$. Then the biting lemma, Lemma 6.15, provides subsequences, again denoted $\{|u^k|^p\}_{k\in\mathbb{N}}$ and $\{|v^k|^q\}_{k\in\mathbb{N}}$, and sequences $\{B^k\}_{k\in\mathbb{N}}$ and $\{D^k\}_{k\in\mathbb{N}}$ of subsets of $B(0,1)$ with $|B^k|\to 0$ and $|D^k|\to 0$ as $k$ tends to infinity, such that $\{\tilde u^k\}_{k\in\mathbb{N}}$ with $\tilde u^k = u^k\chi_{B(0,1)\setminus B^k}$ and $\{\tilde v^k\}_{k\in\mathbb{N}}$, where $\tilde v^k = v^k\chi_{B(0,1)\setminus D^k}$, are equi-integrable sequences in $L^p$ and $L^q$, respectively. Our goal is now to show that all the assumptions on $\{u^k\}_{k\in\mathbb{N}}$ and $\{v^k\}_{k\in\mathbb{N}}$ carry over to $\{\tilde u^k\}_{k\in\mathbb{N}}$ and $\{\tilde v^k\}_{k\in\mathbb{N}}$. Since $\{\tilde u^k\}_{k\in\mathbb{N}}$ and $\{\tilde v^k\}_{k\in\mathbb{N}}$ are uniformly bounded in $L^p(B(0,1);\mathbb{R}^n)$ and $L^q(B(0,1);\mathbb{R}^n)$, respectively, we obtain after extracting a subsequence that $\tilde u^k\rightharpoonup\tilde u$ in $L^p(B(0,1);\mathbb{R}^n)$ and $\tilde v^k\rightharpoonup\tilde v$ in $L^q(B(0,1);\mathbb{R}^n)$. Moreover, it holds by Hölder's inequality that

$$\|\tilde u^k - u^k\|_{L^1(B(0,1);\mathbb{R}^n)} = \|u^k\|_{L^1(B^k;\mathbb{R}^n)} \le |B^k|^{1/q}\|u^k\|_{L^p(B(0,1);\mathbb{R}^n)}\to 0 \tag{6.63}$$

as $k\to\infty$. Analogously one can show

$$\tilde v^k - v^k \to 0 \quad \text{in} \quad L^1(B(0,1);\mathbb{R}^n). \tag{6.64}$$

These findings finally imply $\tilde u = u$ and $\tilde v = v$. Further, we infer from (6.63) that $\operatorname{div} u^k - \operatorname{div}\tilde u^k = \operatorname{div}(\tilde u^k - u^k) \xrightarrow{\square} 0$ in $W^{-1,1}(B(0,1))$ . By (6.64) one has $(\operatorname{curl} v^k)_{12} - (\operatorname{curl}\tilde v^k)_{12} = \operatorname{div}\big((v^k-\tilde v^k)_2, -(v^k-\tilde v^k)_1, 0,\dots,0\big) \xrightarrow{\square} 0$ in $W^{-1,1}(B(0,1))$ and similarly for $(\operatorname{curl} v^k)_{ij}$ with $i,j\in\{1,\dots,n\}$. So it has just been shown that for $k\to\infty$

$$\operatorname{div}\tilde u^k \xrightarrow{\square} 0 \quad \text{in } W^{-1,1}(B(0,1)) \quad \text{and} \quad \operatorname{curl}\tilde v^k \xrightarrow{\square} 0 \quad \text{in } W^{-1,1}(B(0,1);\mathbb{R}^{n\times n}).$$

*Step 3: Application of the classical div-curl lemma.*
By Step 2 the sequences $\{\tilde u^k\}_{k\in\mathbb{N}}$ and $\{\tilde v^k\}_{k\in\mathbb{N}}$ meet all the requirements necessary for the application of Lemma 6.22 and 6.23, respectively. Thus, for every $V\subset\subset B(0,1)$ we have

a sequence $\{\tilde{a}^k\}_{k\in\mathbb{N}} \subset L^p(V;\mathbb{R}^n)$ with $\tilde{a}^k \rightharpoonup u$ in $L^p(V;\mathbb{R}^n)$ and $\operatorname{div}\tilde{a}^k = 0$ in $W^{-1,p}(V)$ for all $k\in\mathbb{N}$, so that $\{\operatorname{div}\tilde{a}_k\}_{k\in\mathbb{N}}$ is obviously compact in $W^{-1,p}(V)$. Furthermore there is $\{\tilde{b}^k\}_{k\in\mathbb{N}}$ with $\tilde{b}^k \rightharpoonup v$ in $L^q(V;\mathbb{R}^n)$ and $\{\operatorname{curl}\tilde{b}^k\}_{k\in\mathbb{N}}$ compact in $W^{-1,q}(V;\mathbb{R}^{n\times n})$, which results from $\operatorname{curl}\tilde{b}^k \to 0$ in $W^{-1,q}(V;\mathbb{R}^{n\times n})$.
So the classical div-curl lemma in its version for general dual exponents mentioned directly below Theorem 6.18 gives $\tilde{a}^k\cdot\tilde{b}^k \rightharpoonup u\cdot v$ in $\mathcal{D}'(B(0,1))$. By Lemma 6.22 and 6.23

$$\tilde{u}^k\cdot\tilde{v}^k - \tilde{a}^k\cdot\tilde{b}^k = \tilde{a}^k\cdot(\tilde{v}^k-\tilde{b}^k) + (\tilde{u}^k-\tilde{a}^k)\cdot\tilde{v}^k \;\longrightarrow 0 \;\text{ in }\; L^1(B(0,1)) \tag{6.65}$$

for $k\to\infty$, since both terms on the right-hand side consist of a scalar product between a uniformly bounded factor and a factor converging strongly to zero in the corresponding dual space. All in all this yields

$$\tilde{u}^k\cdot\tilde{v}^k \rightharpoonup u\cdot v \quad\text{in}\;\; \mathcal{D}'(B(0,1)). \tag{6.66}$$

*Step 4: From $\mathcal{D}'(B(0,1))$-convergence of $\tilde{u}^k\cdot\tilde{v}^k$ to weak-$L^1$ convergence of $u^k\cdot v^k$.*
We recall from the definition of $\tilde{u}^k$ and $\tilde{v}^k$ in Step 2 that $u^k\cdot v^k - \tilde{u}^k\cdot\tilde{v}^k = (u^k\cdot v^k)\chi_{(B^k\cup D^k)}$ for $k\in\mathbb{N}$. The expression on the right-hand side tends to zero in $L^1$ as $k\to\infty$, because $\{u^k\cdot v^k\}_{k\in\mathbb{N}}$ is equi-integrable by assumption and $|B^k\cup D^k| \le |B^k| + |D^k| \to 0$. In view of (6.66) the equi-integrability and uniform $L^1$-boundedness of $\{u^k\cdot v^k\}_{k\in\mathbb{N}}$ lead to $u^k\cdot v^k \rightharpoonup u\cdot v$ in $L^1(B(0,1))$, where $\{u^k\}_{k\in\mathbb{N}}$ and $\{v^k\}_{k\in\mathbb{N}}$ stand for subsequences of the original sequences. Observing that the limit $u\cdot v$ is independent of the chosen subsequence one obtains convergence of the entire sequence. □

### 6.5.3. Applications

Now we state some consequences of the generalized div-curl lemma in Theorem 6.20. To start with, the next corollary contains the solution to the 2D problem that served as a motivation at the beginning of this section.

**Corollary 6.24** *For a bounded Lipschitz domain $\Omega\subset\mathbb{R}^2$ and $p\ge 2$ let $\{u^k\}_{k\in\mathbb{N}} \subset W^{1,1}(\Omega;\mathbb{R}^2)$ be such that $\nabla u^k = A^k + H^k$ with $H^k \to 0$ in $L^1(\Omega;\mathbb{R}^{2\times 2})$ and $A^k \rightharpoonup A$ in $L^p(\Omega;\mathbb{R}^{2\times 2})$. If the sequence $\{\det A^k\}_{k\in\mathbb{N}}$ is equi-integrable, then* $\det A^k \rightharpoonup \det A$ *in $L^{p/2}(\Omega)$ as $k\to\infty$.*

*PROOF.* To put it short the claim follows by applying Theorem 6.20 to the sequences that result from taking the first row and the rotated second row of $A^k$.
At length, it holds $\det A^k = e_1^T A^k \cdot J e_2^T A^k$ with $J$ the counterclockwise rotation by $\pi/2$ in the plane, meaning that $\det A^k$ is the scalar product of the first row of $A^k$ and its rotated second row. Instead of columns we use rows for this representation, because this will allow

us to exploit the gradient structure of $A^k + H^k$ in the following. Next we apply Theorem 6.20 to $\{v^k\}_{k\in\mathbb{N}} = \{e_1^T A^k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^2)$ and $\{w^k\}_{k\in\mathbb{N}} = \{Je_2^T A^k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^2)$. To see that this choice of sequences satisfies the requirements of Theorem 6.20, we observe that (6.55) and $\operatorname{curl}(e_j^T \nabla u^k) = 0$ for $j \in \{1,2\}$ together with $H^k \to 0$ in $L^1(\Omega;\mathbb{R}^{2\times 2})$ imply

$$\operatorname{curl} v^k = \operatorname{curl}(e_1^T A^k) = \operatorname{div}(Je_1^T H^k) \xrightarrow{\square} 0 \quad \text{in} \quad W^{-1,1}(\Omega;\mathbb{R}^{2\times 2})$$

and

$$\operatorname{div} w^k = \operatorname{div}(Je_2^T A^k) = \operatorname{curl}(e_2^T H^k) = -\operatorname{div}(Je_2^T H^k) \xrightarrow{\square} 0 \quad \text{in} \quad W^{-1,1}(\Omega)$$

for $k \to \infty$. Hence, $\det A^k \rightharpoonup \det A$ in $L^1(\Omega)$. Since $\{\det A^k\}_{k\in\mathbb{N}}$ is bounded in $L^{p/2}(\Omega)$, there is a subsequence converging weakly in $L^{p/2}$ to $\det A$ by the uniqueness of the limit. With the limit being independent of the subsequence the whole sequence converges weakly in $L^{p/2}$. $\square$

Here are two more implications which are exactly adapted to our needs regarding the recovery of the determinant constraint in the three-dimensional setting, compare Chapter 8. Let us discuss convergence of cofactors first.

**Corollary 6.25** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $p \geq 2$. Suppose $\{u^k\}_{k\in\mathbb{N}} \subset W^{1,1}(\Omega;\mathbb{R}^3)$ such that $\nabla u^k = A^k + H^k$ with $H^k \to 0$ in $L^1(\Omega;\mathbb{R}^{3\times 3})$ and $A^k \rightharpoonup A$ in $L^p(\Omega;\mathbb{R}^{3\times 3})$. If the sequence $\{\operatorname{cof} A^k\}_{k\in\mathbb{N}}$ is equi-integrable, then $\operatorname{cof} A^k \rightharpoonup \operatorname{cof} A$ in $L^{p/2}(\Omega;\mathbb{R}^{3\times 3})$ as $k \to \infty$.*

*PROOF.* It is sufficient to consider $(\operatorname{cof} A^k)_{33}$, because the arguments for the other entries are in perfect analogy. We write

$$(\operatorname{cof} A^k)_{33} = \big((A^k)_{11}, (A^k)_{12}, (A^k)_{13}\big) \cdot \big((A^k)_{22}, -(A^k)_{21}, 0\big) = v^k \cdot w^k,$$

where $v^k, w^k \in L^p(\Omega;\mathbb{R}^3)$ by assumption. Then,

$$\operatorname{curl} v^k = \operatorname{curl}(e_1^T A^k) = \operatorname{curl}(e_1^T \nabla u^k - e_1^T H^k) = -\operatorname{curl}(e_1^T H^k)$$

and

$$\operatorname{div} w^k = \operatorname{curl}(e_2^T A^k) = -\operatorname{curl}(e_2^T H^k)$$

$\square$-converge to zero in $W^{-1,1}$, so that we can apply the div-curl lemma of Theorem 6.20 to derive $(\operatorname{cof} A^k)_{33} = v^k \cdot w^k \rightharpoonup (A_{11}, A_{12}, A_{13}) \cdot (A_{22}, -A_{21}, 0) = (\operatorname{cof} A)_{33}$ in $L^1(\Omega)$. Finally, weak convergence in $L^{p/2}$ follows as in the proof of Lemma 6.24. $\square$

To conclude this section let us give one more consequence of Theorem 6.20 dealing with the convergence of determinants in the special 3D situation of Chapter 8. Here we proceed by applying the generalized div-curl lemma to the rows of the matrices and their cofactors.

**Corollary 6.26** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $p \geq 2$ and $q \geq \frac{2p}{p-1}$. Suppose $\{u^k\}_{k\in\mathbb{N}} \subset W^{1,1}(\Omega;\mathbb{R}^3)$ such that $\nabla u^k = A^k + H^k$ with*

$$\begin{array}{llll} H^k e_1 \to 0 & \text{in } L^q(\Omega;\mathbb{R}^3), & A^k e_1 \overset{*}{\rightharpoonup} A e_1 & \text{in } L^\infty(\Omega;\mathbb{R}^3), \\ H^k e_2 \to 0 & \text{in } L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^3), & A^k e_2 \rightharpoonup A e_2 & \text{in } L^{\min\{p,q\}}(\Omega;\mathbb{R}^3), \\ H^k e_3 \to 0 & \text{in } L^q(\Omega;\mathbb{R}^3), & A^k e_3 \overset{*}{\rightharpoonup} A e_3 & \text{in } L^\infty(\Omega;\mathbb{R}^3) \end{array} \tag{6.67}$$

*as $k \to \infty$ for some $A \in \mathbb{R}^{3\times 3}$. Then* $\det A^k \rightharpoonup \det A$ *in $L^1(\Omega)$.*

*PROOF.* Here we exploit that $\det A^k$ can be expressed as

$$\det A^k = e_3^T \operatorname{cof} A^k \cdot e_3^T A^k.$$

Defining $v^k = e_3^T A^k$ and $w^k = e_3^T \operatorname{cof} A^k = e_1^T A^k \wedge e_2^T A^k$ for $k \in \mathbb{N}$ we have by assumption that $v^k \rightharpoonup e_3^T A$ in $L^{\min\{p,q\}}(\Omega;\mathbb{R}^3)$ and the application of Corollary 6.25 to $\{w^k\}_{k\in\mathbb{N}}$ yields $w^k \rightharpoonup e_3^T \operatorname{cof} A$ in $L^{\min\{p,q\}/2}(\Omega;\mathbb{R}^3)$. By the special structure of $A^k$, see (6.67), one even gets $w^k \rightharpoonup e_3^T \operatorname{cof} A$ in $L^{\min\{p,q\}}(\Omega;\mathbb{R}^3)$. Notice that $\{\det A^k\}_{k\in\mathbb{N}} = \{v^k \cdot w^k\}_{k\in\mathbb{N}}$ is equi-integrable, for it is uniformly bounded in $L^{\min\{p,q\}}(\Omega)$ and $\min\{p,q\} \geq 2$. Moreover, it holds

$$(\operatorname{curl} v^k)_{12} = \Big(\operatorname{curl}(e_3^T A^k)\Big)_{12} = -\operatorname{div}\big((H^k)_{32}, -(H^k)_{31}, 0\big) \xrightarrow{\square} 0$$

in $W^{-1,1}(\Omega)$. For the other entries of $\operatorname{curl} v^k$ we use similar arguments. As a next step the convergence behavior of $\operatorname{div} w^k$ has to be studied. Here, $\operatorname{div}(e_3^T \operatorname{cof} \nabla u^k) = 0$ leads to

$$\begin{aligned} \operatorname{div} w^k &= \operatorname{div}(e_3^T \operatorname{cof} A^k - e_3^T \operatorname{cof} \nabla u^k) = \operatorname{div}(e_1^T A^k \wedge e_2^T A^k - e_1^T \nabla u^k \wedge e_2^T \nabla u^k) \\ &= \operatorname{div}(e_1^T A^k \wedge e_2^T (A^k - \nabla u^k) + e_1^T (A^k - \nabla u^k) \wedge e_2^T \nabla u^k) \\ &= -\operatorname{div}(e_1^T A^k \wedge e_2^T H^k + e_1^T H^k \wedge e_2^T A^k + e_1^T H^k \wedge e_2^T H^k) \end{aligned} \tag{6.68}$$

for $k \in \mathbb{N}$. With the inclusions $L^p(\Omega) \cdot L^q(\Omega) \subset L^{\frac{pq}{p+q}}(\Omega)$, $L^q(\Omega) \cdot L^q(\Omega) \subset L^{q/2}(\Omega)$ and $L^q(\Omega) \cdot L^{\frac{pq}{p+q}}(\Omega) \subset L^{\frac{pq}{2p+q}}(\Omega)$ following by Hölder's inequality, we deduce from (6.67) that

$$\begin{array}{lll} e_1^T A^k \wedge e_2^T H^k \to 0 & \text{in} & L^{\min\{\frac{pq}{p+q}, \frac{q}{2}\}}(\Omega;\mathbb{R}^3), \\ e_1^T H^k \wedge e_2^T A^k \to 0 & \text{in} & L^{\min\{\frac{pq}{p+q}, \frac{q}{2}\}}(\Omega;\mathbb{R}^3), \\ e_1^T H^k \wedge e_2^T H^k \to 0 & \text{in} & L^{\frac{pq}{2p+q}}(\Omega;\mathbb{R}^3). \end{array} \tag{6.69}$$

In view of the properties of $p$ and $q$, which imply $\frac{pq}{2p+q} \geq 1$, $\frac{q}{2} \geq 1$ and $\frac{pq}{p+q} \geq 1$, one actually has convergence in $L^1(\Omega;\mathbb{R}^3)$ for the three expressions of (6.69). This implies $\operatorname{div} w^k \xrightarrow{\square} 0$ in $W^{-1,1}(\Omega)$, so that all requirements of Theorem 6.20 are fulfilled and we finally obtain $\det A^k = w^k \cdot v^k \rightharpoonup e_3^T \operatorname{cof} A \cdot e_3^T A = \det A$ in $L^1(\Omega)$, as asserted. $\square$

# 7. Results for the two-dimensional setting

The following chapter is the heart of this thesis. Here we prove the main results for the 2D single-slip models with elastic energy as they were pointed out in the introduction. This means we show soft material behavior for the model without hardening, while an asymptotic analysis of the system with linear hardening will reveal elastically rigid limiting behavior. Moreover, various generalizations of these findings are discussed. These include general growth exponent and more realistic elastic energies.
Before we make a start on this, let us briefly recall some of the basic facts from Chapter 4. The condensed energy density with general elastic and plastic growth exponents reads

$$W_\varepsilon(F) = W_{\varepsilon;q,p}(F) = \inf_{\gamma\in\mathbb{R}}\Big\{\frac{1}{\varepsilon}\operatorname{dist}^q\big(F(\mathbb{I}-\gamma s\otimes m),\mathrm{SO}(2)\big) + |\gamma|^p\Big\} \tag{7.1}$$

for $F \in \mathbb{R}^{2\times 2}$, where $\varepsilon > 0$ measures the penalization of elastic strain. Concerning the active slip system $(s, m)$ we assume from now on that $m = s^\perp$. The parameter $q \geq 1$ is the elastic growth exponent and if not stated otherwise, the plastic one $p$ satisfies $p \in \{1\} \cup [2,\infty)$, since these are exactly the cases where the relaxations of the corresponding elastically rigid models are known explicitly. By Theorem 4.1, if no hardening is involved, i.e. $p = 1$, we have

$$W_1^{\mathrm{qc}}(F) = \begin{cases} \sqrt{|F|^2-2}, & \text{if } F \in \mathcal{N}^{(2)},\\ \infty, & \text{otherwise}\end{cases} \tag{7.2}$$

and for $p \geq 2$ it holds

$$W_p^{\mathrm{qc}}(F) = \begin{cases} (|Fm|^2-1)^{p/2}, & \text{if } F \in \mathcal{N}^{(2)},\\ \infty, & \text{otherwise,}\end{cases} \tag{7.3}$$

where $\mathcal{N}^{(2)} = \big\{F \in \mathbb{R}^{2\times2}|\det F = 1, |Fs| \leq 1\big\}$. Besides, $W_p^{\mathrm{pc}}(F) = W_p^{\mathrm{rc}}(F) = W_p^{\mathrm{qc}}(F)$ for all $F \in \mathbb{R}^{2\times 2}$ and $p \in \{1\} \cup [2,\infty)$.

## 7.1. Fundamental properties of the condensed energy density

To begin with let us state a lemma summarizing some obvious but important properties of the energy density $W_\varepsilon$.

**Lemma 7.1** *The energy density $W_\varepsilon$ of* (7.1) *is non-negative, finite and frame-indifferent, i.e. $W_\varepsilon(F) = W_\varepsilon(RF)$ for all $R \in \mathrm{SO}(2)$ and $F \in \mathbb{R}^{2\times 2}$. Further, $W_\varepsilon(R) = 0$ for all $R \in \mathrm{SO}(2)$.*

It will turn out convenient for future calculations and estimates to use the following representation of $W_\varepsilon = W_{\varepsilon;q,p}$ with $q \geq 2$, which highlights its special structure with respect to the directions $s$ and $m$.
Note that every $Q \in \mathrm{SO}(2)$ can be expressed as $Q = a \otimes s + a^\perp \otimes m$ with $a \in \mathbb{S}^1$, since $(s, m)$ is an orthonormal basis in $\mathbb{R}^2$. Then we write

$$
\begin{aligned}
W_\varepsilon(F) &= \min_{\gamma\in\mathbb{R}, Q\in \mathrm{SO}(2)} \frac{1}{\varepsilon}\left(|F(\mathbb{I} - \gamma s\otimes m) - Q|\right)^q + |\gamma|^p \\
&= \min_{\gamma\in\mathbb{R}, a\in\mathbb{S}^1} \frac{1}{\varepsilon}\left(\left|(Fs - a)\otimes s + (Fm - \gamma Fs - a^\perp)\otimes m\right|\right)^q + |\gamma|^p \\
&= \min_{\gamma\in\mathbb{R}, a\in\mathbb{S}^1} \frac{1}{\varepsilon}\left(|Fs - a|^2 + |Fm - \gamma Fs - a^\perp|^2\right)^{q/2} + |\gamma|^p \\
&\geq \min_{\gamma\in\mathbb{R}, a\in\mathbb{S}^1} \frac{1}{\varepsilon}\left(|Fs - a|^q + |Fm - \gamma Fs - a^\perp|^q\right) + |\gamma|^p \\
&= \frac{1}{\varepsilon}\left(|Fs - a_\varepsilon(F)|^q + |Fm - \gamma_\varepsilon(F)Fs - a_\varepsilon^\perp(F)|^q\right) + |\gamma_\varepsilon(F)|^p, \qquad (7.4)
\end{aligned}
$$

where $a_\varepsilon(F) \in \mathbb{S}^1$ and $\gamma_\varepsilon(F) \in \mathbb{R}$ are defined by the last inequality. For simplicity of notation we will leave out the arguments in the expressions $a_\varepsilon(F)$ and $\gamma_\varepsilon(F)$ and denote them by $a_\varepsilon$ and $\gamma_\varepsilon$ instead.
The inequality

$$
(y+z)^{q/2} \geq (y^{q/2} + z^{q/2}) \qquad (7.5)
$$

for all $z, y \geq 0$ and $q \geq 2$, which was used in (7.4), follows from this simple calculation. By computing the derivatives of $f, g : [0,\infty) \to \mathbb{R}$ given by $f(x) = (1 + x^{q/2})^{2/q}$ with $q > 2$ and $g(x) = x + 1$ it is immediate that $f \leq g$ on $[0,\infty)$, since $g(0) = f(0)$. For $y, z > 0$ we set $x = \frac{y}{z}$, so that $\left(1 + \frac{y^{q/2}}{z^{q/2}}\right)^{2/q} \leq 1 + \frac{y}{z}$. This is equivalent to $(y+z)^{q/2} \geq (y^{q/2} + z^{q/2})$. If $y = 0$, $z = 0$ or $q = 2$, there is nothing to show.

### 7.1.1. Algebraic estimates

We start the investigation of the energy density in (7.1) by studying its growth behavior both from above and below. Indeed, this is one of the crucial points of the whole problem, since $W_\varepsilon$ shapes up as a density with non-standard growth.

**Lemma 7.2 (Growth and coercivity properties of $W_\varepsilon$)** *For $\varepsilon > 0$, $q \geq 2$ and $F \in \mathbb{R}^{2\times 2}$ there are the upper estimates* *(i)* $W_\varepsilon(F) \leq \frac{2^{(q-1)}}{\varepsilon}(|F|^q + 2^{q/2})$,

*(ii)* $W_\varepsilon(F) \leq W(F)$

*and the lower bounds*

*(iii)* $\big||Fs| - 1\big| \leq \big(\varepsilon W_\varepsilon(F)\big)^{1/q}$,

*(iv)* $|Fm| \leq \varepsilon^{1/q} W_\varepsilon(F)^{\frac{p+q}{pq}} + \varepsilon^{1/q} W_\varepsilon(F)^{1/q} + W_\varepsilon(F)^{1/p} + 1$.

**Remark 7.3** *1. This lemma reveals the anisotropic growth and coercivity behavior of the condensed energy density $W_\varepsilon$. Indeed, $W_\varepsilon$ features q-th order growth from below with respect to the component $Fs$, while we only get growth of order $\frac{pq}{p+q}$ for $Fm$.*
*2. Notice that there exist $p, q \in [1, \infty)$ such that $\frac{pq}{p+q} < 1$, just take $p = 1$ and $q = 2$ as an example. Then, there are certain directions along which $W_\varepsilon$ grows merely sublinearly. For instance, considering the curve $t \mapsto F_t = (t\mathbb{I})(\mathbb{I} + t^{q/p} s \otimes m)$ gives*

$$W_{\varepsilon;q,p}(F_t) \leq \frac{c}{\varepsilon}|F_t|^{\frac{pq}{p+q}} \tag{7.6}$$

*for sufficiently large $t > 0$ with a constant $c > 0$, compare [20]. This can be seen as follows. With the multiplicative decomposition $F_t = (t\mathbb{I})(\mathbb{I} + t^{q/p} s \otimes m) = (F_{\mathrm{el}})_t (F_{\mathrm{pl}})_t$ we observe for large $t > 0$ that $|F_t| \sim t^{\frac{p+q}{p}}$, $|(F_{\mathrm{el}})_t| \sim t \sim |F_t|^{\frac{p}{p+q}}$ and $|(F_{\mathrm{pl}})_t| \sim t^{\frac{q}{p}} \sim |F_t|^{\frac{q}{p+q}}$. Consequently,*

$$W_{\varepsilon;q,p}(F_t) \leq \frac{1}{\varepsilon}\operatorname{dist}^q\big((F_{\mathrm{el}})_t, SO(2)\big) + |t^{q/p}|^p \sim \frac{c}{\varepsilon}\big(|(F_{\mathrm{el}})_t|^q + t^q\big) \sim \frac{c}{\varepsilon}|F_t|^{\frac{pq}{p+q}},$$

*which implies* (7.6).
*3. Apart from anisotropy and sublinearity there is another issue of non-standard growth concerned with the size of the gap between the upper and lower growth exponents of $W_\varepsilon$. In a more general context the variational problem with energy density $W_\varepsilon$ can be regarded as a problem of $(p, q)$-growth [33, 51, 46] as it is called in the literature. By this notion one characterizes variational problems whose growth and coercivity conditions are satisfied with different exponents p and q, respectively. In order to avoid confusion with the elastic and plastic material parameters of $W_\varepsilon$ we rather refer to $(p, q)$-growth as $(r, t)$-growth in the following. Assume $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ is quasiconvex with*

$$c_1|F|^r - c_2 \leq f(F) \leq C(|F|^t + 1)$$

*for all $F \in \mathbb{R}^{n\times n}$ with constants $c_1, c_2, C > 0$ and $1 < r < t < \infty$. Then, the application of the classical theory on relaxation and lower semicontinuity of functionals with $(r, t)$-growth requires a relation between $r$ and $t$, namely $\frac{t}{r} < \frac{n}{n-1}$, see [33, 46]. If we take for example our single-slip model with forth order elastic energy and linear hardening* *and consider $f = W_{\varepsilon;4,2}$, Lemma 7.2 yields $r = 4/3$ and $t = 4$ and therefore we have $\frac{t}{r} = 3 > \frac{2}{2-1}$ with $n = 2$, so that the above stated postulate is not fulfilled here. For $W_{\varepsilon;2,2}$ we do not even have $r > 1$. This suggests that the general approach will not help in the present context and that a situational treatment exploiting specific information on $W_\varepsilon$ will be necessary.*

*PROOF.* For the upper bound (i) we estimate by choosing $\gamma = 0$ in (7.1),

$$W_\varepsilon(F) \leq \frac{1}{\varepsilon}\operatorname{dist}^q(F, \mathrm{SO}(2)) \leq \frac{1}{\varepsilon}(2|F|^2+4)^{q/2} \leq \frac{2^{(q-1)}}{\varepsilon}(|F|^q + 2^{q/2}).$$

As to (ii) it is sufficient in view of $W(F) = \infty$ for all $F \in \mathbb{R}^{2\times 2} \setminus \mathcal{M}^{(2)}$ to restrict to $F \in \mathcal{M}^{(2)}$ written as $F = R(\mathbb{I} + \tau s \otimes m)$ with $R \in \mathrm{SO}(2)$ and $\tau \in \mathbb{R}$, compare (4.3). Then,

$$W_\varepsilon(F) = \inf_{\gamma\in\mathbb{R}}\Big\{\frac{1}{\varepsilon}\operatorname{dist}^q\big(\mathbb{I} - (\gamma-\tau)s\otimes m, \mathrm{SO}(2)\big) + |\gamma|^p\Big\} \leq |\tau|^p = W(F).$$

Using (7.4) inequality (iii) follows from

$$\big||Fs|-1\big| = \big||Fs|-|a_\varepsilon|\big| \leq |Fs - a_\varepsilon| \leq (\varepsilon W_\varepsilon(F))^{1/q}.$$

Further let us compute

$$\begin{aligned}
|Fm| &\leq \big|Fm - \gamma_\varepsilon Fs - a_\varepsilon^\perp\big| + |\gamma_\varepsilon|\,|Fs| + |a_\varepsilon^\perp| \\
&\leq (\varepsilon W_\varepsilon(F))^{1/q} + W_\varepsilon(F)^{1/p}\big((\varepsilon W_\varepsilon(F))^{1/q}+1\big)+1 \\
&\leq \varepsilon^{1/q} W_\varepsilon(F)^{\frac{p+q}{pq}} + \varepsilon^{1/q}W_\varepsilon(F)^{1/q} + W_\varepsilon(F)^{1/p} + 1\,.
\end{aligned}$$

This is (iv). □

The next lemma deals with an a priori estimate involving determinants.

**Lemma 7.4** *Let $\varepsilon \in (0,1)$, $q \geq 2$ and $F \in \mathbb{R}^{2\times 2}$. Then*

$$|\det F - 1| \leq \varepsilon^{1/q}\big(W_\varepsilon(F)^{2/q}+1\big).$$

*PROOF.* Here we exploit the representation of $\det F$ in the form $\det F = Fs\cdot JFm$, where $J$ is the counterclockwise rotation by $\pi/2$ in the plane. Then one calculates

$$\begin{aligned}
\det F &= Fs\cdot JFm = Fs\cdot J(Fm - \gamma_\varepsilon Fs) \\
&= (Fs - a_\varepsilon)\cdot J(Fm-\gamma_\varepsilon Fs) + a_\varepsilon \cdot J(Fm - \gamma_\varepsilon Fs) \\
&= (Fs-a_\varepsilon)\cdot J(Fm - \gamma_\varepsilon Fs - a_\varepsilon^\perp) + (Fs - a_\varepsilon)\cdot Ja_\varepsilon^\perp + a_\varepsilon\cdot J(Fm-\gamma_\varepsilon Fs - a_\varepsilon^\perp) + 1\,.
\end{aligned}$$

By (7.4) there are the estimates

$$\begin{aligned} |Fs - a_\varepsilon| + |Fm - \gamma_\varepsilon Fs - a_\varepsilon^\perp| &\le \sqrt{2}(\varepsilon W_\varepsilon(F))^{1/q}\,, \\ |Fs - a_\varepsilon|\,|Fm - \gamma_\varepsilon Fs - a_\varepsilon^\perp| &\le \frac{1}{2}(\varepsilon W_\varepsilon(F))^{2/q}\,, \end{aligned}$$

which yield

$$|\det F - 1| \le \varepsilon^{1/q}\left(\frac{1}{2}W_\varepsilon(F)^{2/q} + \sqrt{2}W_\varepsilon(F)^{1/q}\right) \le \varepsilon^{1/q}(W_\varepsilon(F)^{2/q} + 1).$$

In the last step we used $\sqrt{2W_\varepsilon(F)^{2/q}} \le \frac{1}{2}W_\varepsilon^{2/q}(F) + 1$. This completes the proof. $\square$

This paragraph is concluded by discussing the dependence of $W_\varepsilon$ on $\varepsilon$. In the second lemma we focus in particular on the limit of small $\varepsilon$.

**Lemma 7.5 (Monotonicity of $W_\varepsilon$ with respect to $\varepsilon$)** *If $0 < \varepsilon \le \widetilde{\varepsilon}$, then $W_\varepsilon(F) \ge W_{\widetilde{\varepsilon}}(F)$ for all $F \in \mathbb{R}^{2\times 2}$.*

*PROOF.* This observation is an immediate consequence of the definition of $W_\varepsilon$. $\square$

**Lemma 7.6 (Pointwise convergence of $W_\varepsilon$)** *The rigid energy density $W$ is the pointwise limit of $W_\varepsilon$ as $\varepsilon \to 0$, i.e. $\lim_{\varepsilon\to 0} W_\varepsilon(F) = W(F)$ for all $F \in \mathbb{R}^{2\times 2}$.*

*PROOF.* For $F \in \mathbb{R}^{2\times 2} \setminus \mathcal{M}^{(2)}$ the convergence $\lim_{\varepsilon\to 0} W_\varepsilon(F) = \infty$ follows immediately from Lemma 7.2 (iii) and Lemma 7.4. So we only need to consider $F \in \mathcal{M}^{(2)}$, which can be written as $F = R(\mathbb{I} + \tau s \otimes m)$ with $R \in \mathrm{SO}(2)$ and $\tau \in \mathbb{R}$. Then,

$$\begin{aligned} W_\varepsilon(F) &= \min_{\gamma\in\mathbb{R}}\Big\{\frac{1}{\varepsilon}\operatorname{dist}^q\big(\mathbb{I} - (\gamma - \tau)s \otimes m, \mathrm{SO}(2)\big) + |\gamma|^p\Big\} \\ &= \frac{1}{\varepsilon}\operatorname{dist}^q\big(\mathbb{I} - (\gamma_\varepsilon(\tau) - \tau)s \otimes m, \mathrm{SO}(2)\big) + |\gamma_\varepsilon(\tau)|^p, \end{aligned}$$

where $\gamma_\varepsilon(\tau)$ is defined by the last equality. Regarding Lemma 7.2 (ii) we learn that $\operatorname{dist}^q\big(\mathbb{I} - (\gamma_\varepsilon(\tau) - \tau)s \otimes m, \mathrm{SO}(2)\big) \le \varepsilon W(F)$ for all $\varepsilon > 0$. As a consequence $\gamma_\varepsilon(\tau) \to \tau$ as $\varepsilon$ tends to zero. Hence, $\lim_{\varepsilon\to 0} W_\varepsilon(F) = |\tau|^p = W(F)$. $\square$

### 7.1.2. Estimates for the envelopes

The estimates resulting from Section 7.1.1 can be used to establish polyconvex lower bounds on the envelopes of $W_\varepsilon$.

**Corollary 7.7** *For $\varepsilon \in (0,1)$, $q \ge 2$ and $F \in \mathbb{R}^{2\times 2}$ there are the estimates*

*(i)* $W_\varepsilon^{\mathrm{pc}}(F) \ge \frac{1}{\varepsilon}\max\{|Fs| - 1, 0\}^q$,

*(ii)* $W_\varepsilon^{\mathrm{pc}}(F) \geq \max\{\frac{1}{\varepsilon^{1/q}}|\det F - 1| - 1, 0\}^{q/2}$.

*Moreover, for all* $F \in \mathbb{R}^{2\times 2} \setminus \mathcal{N}^{(2)}$ *one obtains*

$$\lim_{\varepsilon\to 0} W_\varepsilon^{\mathrm{pc}}(F) = W^{\mathrm{pc}}(F) = \infty.$$

*Exactly the same assertions hold true for the quasiconvex and rank-one convex envelopes.*

*PROOF.* The inequalities (i) and (ii) are a direct implication of Lemma 7.2 (iii) and Lemma 7.4, respectively. Indeed, the restriction to positive values turns the right-hand sides into polyconvex lower bounds of $W_\varepsilon$. The pointwise convergence of $W_\varepsilon^{\mathrm{pc}}$ to $\infty$ follows immediately from (i) and (ii). The analogous results for the quasiconvex and rank-one convex envelopes are a consequence of $W_\varepsilon^{\mathrm{pc}} \leq W_\varepsilon^{\mathrm{qc}} \leq W_\varepsilon^{\mathrm{rc}}$, see (3.4). □

### 7.1.3. Energy functionals

Next we introduce the energy functionals corresponding to the condensed energy density $W_\varepsilon$ and the relaxed rigid density $W^{\mathrm{qc}}$. For $\varepsilon > 0$ and $p, q \geq 2$ let the energy functionals $E_\varepsilon$ and $E$ be defined by

$$E_\varepsilon[u] = E_{\varepsilon;q,p}[u] = \begin{cases} \displaystyle\int_\Omega W_{\varepsilon;q,p}(\nabla u)\,\mathrm{d}x, & \text{if } u \in W^{1,\frac{pq}{p+q}}(\Omega;\mathbb{R}^2), \\ \infty, & \text{otherwise} \end{cases} \tag{7.7}$$

and

$$E[u] = E_p[u] = \begin{cases} \displaystyle\int_\Omega W_p^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x, & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^2),\ \nabla u \in \mathcal{N}^{(2)} \text{ } a.e. \text{ in } \Omega, \\ \infty, & \text{otherwise.} \end{cases} \tag{7.8}$$

In Section 7.3.2 and Section 7.4.1 the relation between these energies is investigated both for $p = q = 2$ and for general growth exponents by proving $\Gamma$-convergence of $E_\varepsilon$ to $E$. This objective in mind some further notation is needed to capture the structure of finite energy deformations. We anticipate the definition of these quantities for general growth exponents for reasons of clarity and to avoid unnecessary repetitions.

Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a bounded energy sequence of $E_\varepsilon$. For $u_\varepsilon \in W^{1,\frac{pq}{p+q}}(\Omega;\mathbb{R}^2)$ we take $a_\varepsilon = a_\varepsilon(\nabla u_\varepsilon) \in \mathbb{S}^1$ and $\gamma_\varepsilon = \gamma_\varepsilon(\nabla u_\varepsilon) \in \mathbb{R}$ as in (7.4) and set

$$\begin{aligned} h_\varepsilon &= (\nabla u_\varepsilon)s - a_\varepsilon\,, \\ f_\varepsilon &= \gamma_\varepsilon h_\varepsilon = \gamma_\varepsilon\big((\nabla u_\varepsilon)s - a_\varepsilon\big)\,, \\ g_\varepsilon &= (\nabla u_\varepsilon)m - f_\varepsilon = (\nabla u_\varepsilon)m - \gamma_\varepsilon(\nabla u_\varepsilon)s + \gamma_\varepsilon a_\varepsilon = g_\varepsilon^{(1)} + g_\varepsilon^{(2)}, \end{aligned} \tag{7.9}$$

where $g_\varepsilon^{(1)} = (\nabla u_\varepsilon)m - \gamma_\varepsilon(\nabla u_\varepsilon)s - a_\varepsilon^\perp$ and $g_\varepsilon^{(2)} = a_\varepsilon^\perp + \gamma_\varepsilon a_\varepsilon$.

Then, there is the additive decomposition

$$\nabla u_\varepsilon = A_\varepsilon + H_\varepsilon \tag{7.10}$$

with $A_\varepsilon = a_\varepsilon \otimes s + g_\varepsilon \otimes m$ and $H_\varepsilon = h_\varepsilon \otimes s + f_\varepsilon \otimes m$.

## 7.2. The model without hardening and quadratic elastic energy

Let us start our investigation of concrete systems with a model without self-hardening and quadratic elastic energy. In terms of condensed energy densities this means we consider (7.1) with $p = 1$ and $q = 2$. Unless stated otherwise we assume exactly this choice of elastic and plastic growth exponents throughout the whole section.
The surprising finding we are going to reveal here results from the sublinear growth of $W_\varepsilon$ pointed out in Remark 7.3. Indeed, the relaxation of $W_\varepsilon$ vanishes completely on the set $\mathcal{N}^{(2)}$, which can be interpreted as follows. On a macroscopic scale, absence of hardening leads to very soft behavior of material samples in response to a large class of applied deformations.

### 7.2.1. Investigation of the polyconvex envelope

Here is our first observation concerning the polyconvex envelope of $W_\varepsilon = W_{\varepsilon;2,1}$.

**Theorem 7.8** *Let $\varepsilon > 0$. Then the polyconvex envelope of $W_\varepsilon = W_{\varepsilon;2,1}$ vanishes on $\mathcal{N}^{(2)}$, in formulas*

$$W_\varepsilon^{\mathrm{pc}}(F) = 0$$

*for all $F \in \mathcal{N}^{(2)} = \{F \in \mathbb{R}^{2\times 2} \mid \det F = 1, \, |Fs| \leq 1\}$.*

**Remark 7.9** *Actually, this result can be understood as an immediate corollary of Theorem 7.10 in view of $W_\varepsilon^{\mathrm{pc}} \leq W_\varepsilon^{\mathrm{rc}}$. Nevertheless we present a direct approach via polyaffine mappings below, since it was the first rigorous proof to back up our conjecture that the envelopes of $W_\varepsilon$ might be trivial on $\mathcal{N}^{(2)}$.*

*PROOF.* The idea of this proof relies on the representation formula given in (3.8), i.e.

$$W_\varepsilon^{\mathrm{pc}}(F) = \sup\{p(F) \mid p : \mathbb{R}^{2\times 2} \to \mathbb{R} \text{ polyaffine with } p \leq W_\varepsilon\}\,. \tag{7.11}$$

Let $p : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be a polyaffine lower bound on $W_\varepsilon$. Our intention is to show $p \leq 0$ on $\mathcal{N}^{(2)}$. By (3.3) there are $A \in \mathbb{R}^{2\times 2}$ and $b, c \in \mathbb{R}$ such that $p(F) = A : F + b \det F + c$ for all $F \in \mathbb{R}^{2\times 2}$. In view of the frame indifference of $W_\varepsilon$, see Lemma 7.1, every polyconvex function of the form $RA : F + b \det F + c$ with $R \in \mathrm{SO}(2)$ is a lower bound on $W_\varepsilon$, if and only if $p$ fulfills this property. Hence we may assume without loss of generality that $A : m \otimes s = 0$ and $A : s \otimes s \geq 0$.
In the following we consider the behavior of $p$ along the curve

$$t \quad \mapsto \quad F(t) = t\mathbb{I} + t^3\, s \otimes m.$$

Notice that $t \mapsto F(t)$ is exactly the curve of sublinear growth pointed out in Remark 7.3. By choosing $\gamma = t^2$ and $a = s$ in the following minimizing process and recalling $m = s^\perp$ one gets

$$\begin{aligned} W_\varepsilon(F(t)) &= \min_{\gamma\in\mathbb{R}, a\in\mathbb{S}^1} \frac{1}{\varepsilon}\left(|F(t)s - a|^2 + |F(t)m - \gamma F(t)s - a^\perp|^2\right) + |\gamma| \\ &\le \frac{1}{\varepsilon}\left(|t\,s - s|^2 + |t\,m + t^3 s - t^2\,t\,s - m|^2\right) + t^2 \\ &= \frac{1}{\varepsilon}\left(|t\,s - s|^2 + |t\,m - m|^2\right) + t^2 = \frac{2}{\varepsilon}(t-1)^2 + t^2, \end{aligned} \tag{7.12}$$

so that $W_\varepsilon(F(t))$ has an upper bound of quadratic growth with respect to $t$. On the other hand, we find with $p \le W_\varepsilon$ on $\mathbb{R}^{2\times 2}$ that for all $t \in \mathbb{R}$

$$\begin{aligned} W_\varepsilon(F(t)) &\ge p(F(t)) = A : F(t) + b\det F(t) + c \\ &= (A : s\otimes m)\,t^3 + b\,t^2 + (A : \mathbb{I})\,t + c. \end{aligned} \tag{7.13}$$

Notice that this lower bound contains a term of order three in $t$. If we compare the asymptotic behavior of (7.13) and (7.12) for both $t \to \infty$ and $t \to -\infty$, we see $A : s\otimes m = 0$. The same reasoning applied to the curve $t \mapsto JF(t)$ with $J$ the counterclockwise rotation by $\pi/2$ in the plane yields $A : m\otimes m = A : Js\otimes m = 0$. Then, for every $F \in \mathcal{N}^{(2)}$ it holds in view of $A : \mathbb{I} = A : (s\otimes s + m\otimes m) = A : s\otimes s \ge 0$ that

$$\begin{aligned} p(F) &= A : F + b\,\det F + c = (A : s\otimes s)(F : s\otimes s) + b\det F + c \\ &= (A : \mathbb{I})(s^T F s) + b\cdot 1 + c \le |Fs|(A : \mathbb{I}) + b\,\det \mathbb{I} + c \\ &\le (A : \mathbb{I}) + b\,\det\mathbb{I} + c = p(\mathbb{I}) \le 0. \end{aligned}$$

The last inequality is due to $W_\varepsilon(\mathbb{I}) = 0$ according to Lemma 7.1 and the requirement $p \le W_\varepsilon$. Summing up we know $p(F) \le 0$ for all $F \in \mathcal{N}^{(2)}$ and every $p$ as in (7.11), so that $W_\varepsilon^{pc} \le 0$ on $\mathcal{N}^{(2)}$. Finally, taking into account that $p \equiv 0$ is polyaffine and a natural lower bound on $W_\varepsilon$ proves the claim. □

### 7.2.2. Relaxation and the rank-one convex envelope

The following result shows that any crystalline state with deformation gradient lying in $\mathcal{N}^{(2)}$ can be well-approximated by microstructures with arbitrarily small energy in the context of our model without hardening. So the material behavior on the macroscopic scale with respect to the considered class of deformations is rather soft due to the formation of fine-scale structure. Mathematically speaking, we determine the relaxation of $W_\varepsilon = W_{\varepsilon;2,1}$ by calculating its quasiconvex envelope. The theorem, which has already been published in [20], reads as follows.

**Theorem 7.10** *Let $\varepsilon > 0$. Then it holds for all $F \in \mathcal{N}^{(2)}$ that*

$$W_\varepsilon^{\mathrm{rc}}(F) = W_\varepsilon^{\mathrm{qc}}(F) = 0.$$

**Remark 7.11** *1. If we recall the representation formula (3.10) for quasiconvex envelopes, Theorem 7.10 says that for all $\varepsilon > 0$, $F \in \mathcal{N}^{(2)}$ and $\Omega \subset \mathbb{R}^2$ bounded with Lipschitz-continuous boundary there exists a sequence $\{u^k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^2)$ with $u^k(x) = Fx$ on $\partial\Omega$ such that*

$$\lim_{k\to\infty} \int_\Omega W_\varepsilon(\nabla u^k)\,\mathrm{d}x = 0.$$

*2. Theorem 7.10 shows that despite of the pointwise convergence $W_\varepsilon \to W$ for $\varepsilon \to 0$ in Lemma 7.6, an analogous convergence cannot hold for the associated quasiconvex envelopes. Indeed, $W_\varepsilon^{\mathrm{qc}}$ is identically zero on the set $\mathcal{N}^{(2)}$, while $W^{\mathrm{qc}}(F)$ is strictly positive for $F \in \mathcal{N}^{(2)} \setminus \mathrm{SO}(2)$, see Theorem 4.1. So in contrast to what one might expect, the model with rigid elasticity does not serve as a good approximation for the system with large elastic energy, since the corresponding relaxations are qualitatively different on the set of relevant matrices.*

*PROOF.* The proof is based on the construction of appropriate rank-one lines. As a motivation for the choice of these lines we refer to Remark 7.2, where it was shown that there are curves featuring sublinear growth with respect to $W_\varepsilon$. Analogously to Theorem 7.8 it is exactly these curves that are the key to the verification of the theorem.
Let $\varepsilon > 0$. We know from (4.6) that $\mathcal{N}^{(2)}$ is the rank-one convex hull of $\mathcal{M}^{(2)}$. Hence, for $F \in \mathcal{N}^{(2)}$ there are rank-one matrices $F^+, F^- \in \mathcal{M}^{(2)}$ and $\lambda \in (0,1)$ such that $F = \lambda F^+ + (1-\lambda)F^-$. Since $W_\varepsilon^{\mathrm{rc}}$ is convex along rank-one lines it holds $W_\varepsilon^{\mathrm{rc}}(F) \leq \lambda W_\varepsilon^{\mathrm{rc}}(F^+) + (1-\lambda)W_\varepsilon^{\mathrm{rc}}(F^-)$. In fact, if the assertion holds for $F \in \mathcal{M}^{(2)}$, it immediately carries over to all $F \in \mathcal{N}^{(2)}$. Therefore it is sufficient to restrict to $\mathcal{M}^{(2)}$-matrices from now on. Due to the frame indifference of $W_\varepsilon$ we can make a further simplification and consider only elements of the form

$$F_* = \mathbb{I} + \sigma\gamma_0 s \otimes m \in \mathcal{M}^{(2)}$$

with $\sigma \in \{-1, 1\}$ and $\gamma_0 > 0$.
At first we construct an auxiliary family of rank-one lines $t \mapsto \widetilde{F}_r(t)$ passing through $\mathbb{I}$ and going close past $F_*$ for $r$ large. If $r > \max\{1, \gamma_0^{1/3}\}$, we set $\widetilde{F}_r(t) = \mathbb{I} + tR_r$ for $t \in \mathbb{R}$ with

$$R_r = (r-1)s \otimes s + \sigma r^3 s \otimes m. \tag{7.14}$$

Taking $t = \gamma_0 r^{-3}$ yields

$$\widetilde{F}_r(\gamma_0 r^{-3}) = \mathbb{I} + \gamma_0\left(\sigma s \otimes m + (r-1)r^{-3} s \otimes s\right) = F_* + \gamma_0(r-1)r^{-3} s \otimes s = F_* + G_r,$$

where $G_r = \gamma_0(r-1)r^{-3}\, s \otimes s$ with $|G_r| \le \gamma_0 r^{-2}$. By a slight modification of $\widetilde{F}_r(t)$ one can find a family of rank-one lines $t \mapsto F_r(t)$ that pass exactly through $F_*$ and are close to $\mathbb{I}$. To this end we set for all $t \in \mathbb{R}$

$$F_r(t) = H_r \widetilde{F}_r(t), \tag{7.15}$$

where $H_r$ is defined by $H_r = F_* \, (F_* + G_r)^{-1}$. Then, $F_r(\gamma_0 r^{-3}) = F_*$ and $F_r(0) = H_r$ with the estimate

$$|F_r(0) - \mathbb{I}| = |H_r - \mathbb{I}| \le |G_r| \, |(F_* + G_r)^{-1}| \le c(\gamma_0)|G_r| \le c(\gamma_0)\, r^{-2}. \tag{7.16}$$

This means $F_r(0)$ is close to the identity matrix for large $r$. Moreover with $\lambda_r = \gamma_0 r^{-3}$ the matrix $F_*$ can be written as

$$F_* = (1 - \lambda_r)F_r(0) + \lambda_r F_r(1).$$

Let us now analyze the energy density $W_\varepsilon$ in $F_r(0)$ and $F_r(1)$. Estimate (7.16) yields

$$W_\varepsilon(F_r(0)) \le W_{\mathrm{el};\varepsilon}(F_r(0)) \le \frac{1}{\varepsilon}\, |F_r(0) - \mathbb{I}|^2 \le c(\gamma_0, \varepsilon) r^{-4}, \tag{7.17}$$

so that $\lim_{r\to\infty} W_\varepsilon(F_r(0)) = 0$. In view of

$$F_r(1)(\mathbb{I} - \sigma r^2 s \otimes m) = H_r(\mathbb{I} + R_r)(\mathbb{I} - \sigma r^2 s \otimes m) = H_r\,(r s \otimes s + m \otimes m)$$

one finds by setting $\gamma = \sigma r^2$ in the definition of $W_\varepsilon$ that

$$W_\varepsilon(F_r(1)) \quad \le \tfrac{2}{\varepsilon}\left(|H_r|^2(r^2+1) + 2\right) + r^2 \le c(\gamma_0, \varepsilon)\big(r^2 + 1\big)\,. \tag{7.18}$$

For the last estimate we used $|H_r|^2 \le |F_*|^2 |(F_* + G_r)^{-1}|^2 \le c(\gamma_0)(2 + \gamma_0^2) = c(\gamma_0)$. Since the rank-one convex envelope of $W_\varepsilon$ is convex along the rank-one line $t \mapsto F_r(t)$ we obtain by (7.18) and (7.17)

$$W_\varepsilon^{\mathrm{rc}}(F_*) \le (1-\lambda_r)W_\varepsilon\big(F_r(0)\big) + \lambda_r W_\varepsilon\big(F_r(1)\big) \le W_\varepsilon\big(F_r(0)\big) + c(\gamma_0, \varepsilon)\frac{r^2+1}{r^3} \longrightarrow 0$$

as $r \to \infty$. Thus, $W_\varepsilon^{\mathrm{rc}}(F_*) = 0$ and $W_\varepsilon^{\mathrm{qc}}(F_*) = 0$ by (3.4), since $W_\varepsilon$ is finite-valued. $\square$

## 7.3. The model with linear hardening and quadratic elastic energy

This section points out the fundamental difference between the model with linear hardening and the one without hardening studied in Section 7.2. The essential finding is that material behavior is less soft in the presence of hardening. In contrast to what was observed before, see Remark 7.11, the elastically rigid system is actually a good approximation to the model with elastic energy provided the critical stress is small compared to the elastic constants.

Unless stated otherwise in this section we take $p = q = 2$.

### 7.3.1. Properties of the relaxed energy densities

Now we want to investigate the relaxations of $W_\varepsilon = W_{\varepsilon;2,2}$ for $\varepsilon > 0$. More precisely, we focus on their relation to the effective elastically rigid energy density $W^{\mathrm{qc}} = W_2^{\mathrm{qc}}$ first. Subsequently, there is a paragraph on regularity properties of $W_\varepsilon^{\mathrm{qc}} = W_{\varepsilon;2,2}^{\mathrm{qc}}$.

#### Pointwise convergence of $W_\varepsilon^{\mathrm{qc}}$ to $W^{\mathrm{qc}}$

The first question one has to answer is whether the envelopes of $W_\varepsilon$ are flat on $\mathcal{N}^{(2)}$ just as the ones of $W_{\varepsilon;2,1}$ are for the system without hardening, see Theorem 7.8. This, however, is not the case as we see from the following result.

**Theorem 7.12** *If $F \in \mathbb{R}^{2\times 2}$, then*

$$\lim_{\varepsilon\to 0} W_\varepsilon^{\mathrm{qc}}(F) = W^{\mathrm{qc}}(F).$$

**Remark 7.13** *1. Note that with literally the same proof Theorem 7.12 remains correct, if $W_\varepsilon^{\mathrm{qc}}$ is replaced by the rank-one or polyconvex envelopes $W_\varepsilon^{\mathrm{rc}}$ and $W_\varepsilon^{\mathrm{pc}}$.*
*2. In particular, the proof yields an explicit lower bound on the quasiconvex envelope of $W_\varepsilon$, which shows $W_\varepsilon^{\mathrm{qc}}(F) > 0$ for all $F \in \mathbb{R}^{2\times 2} \setminus \mathrm{SO}(2)$ and $\varepsilon > 0$. Hence, in presence of linear hardening there is no trivial relaxation as it was encountered in the models without hardening.*

Before we start giving the proof, which is based on elementary estimates and explicit constructions of lower bounds, let us state the subsequent lemma. It contains refined estimates for $W_\varepsilon$ in the special case $p = q = 2$. Depending on whether $\varepsilon$ is small or large one obtains quadratic or linear growth with respect to the component $Fm$.

**Lemma 7.14** *For all $\varepsilon > 0$ and $F \in \mathbb{R}^{2\times 2}$ it holds $W_\varepsilon(F) \geq h_\varepsilon(|Fm|)$ with*

$$h_\varepsilon : \mathbb{R}_0^+ \to \mathbb{R}, \quad z \mapsto h_\varepsilon(z) = \frac{z^2 - \varepsilon z^2 - 1}{1 + \varepsilon z^2}. \tag{7.19}$$

*Moreover, if $|Fm|^2\varepsilon \geq (2+\varepsilon)^2$,*

$$W_\varepsilon(F) \geq \left(\frac{1}{2\sqrt{\varepsilon}}|Fm| - \frac{1}{2}\right). \tag{7.20}$$

*PROOF.* Let $F \in \mathbb{R}^{2\times 2}$ with $|Fm| > 0$ and take $a_\varepsilon$, $\gamma_\varepsilon$ as defined in (7.4). If $\gamma_\varepsilon^2 \geq |Fm|^2$ or $\frac{1}{\varepsilon}|Fs - a_\varepsilon|^2 \geq |Fm|^2$ or $\frac{1}{\varepsilon}|Fm - \gamma_\varepsilon Fs - a_\varepsilon^\perp|^2 \geq |Fm|^2$ we immediately get $W_\varepsilon(F) \geq |Fm|^2 \geq h_\varepsilon(|Fm|)$, which proves the claim in these cases. If none of the upper three inequalities is fulfilled, we estimate

$$\begin{aligned} |Fm|^2 - 1 &= |Fm|^2 - \left(\sqrt{1+\gamma_\varepsilon^2}\right)^2 + \gamma_\varepsilon^2 \\ &= \left(|Fm| - |a_\varepsilon^\perp + \gamma_\varepsilon a_\varepsilon|\right)\left(|Fm| + |a_\varepsilon^\perp + \gamma_\varepsilon a_\varepsilon|\right) + \gamma_\varepsilon^2. \end{aligned}$$

Then, $|a_\varepsilon^\perp + \gamma_\varepsilon a_\varepsilon| > |Fm|$ implies that the product on the right-hand side of the upper equality is negative and consequently $W_\varepsilon(F) \geq \gamma_\varepsilon^2 \geq |Fm|^2 - 1 \geq h_\varepsilon(|Fm|)$. So let us assume from now on that $\left|a_\varepsilon^\perp + \gamma_\varepsilon a_\varepsilon\right| \leq |Fm|$. This implies

$$
\begin{aligned}
&|Fm|^2 - 1 \leq 2|Fm||Fm - a_\varepsilon^\perp - \gamma_\varepsilon a_\varepsilon| + \gamma_\varepsilon^2 \\
&\quad \leq 2|Fm|\left(|Fm - \gamma_\varepsilon Fs - a_\varepsilon^\perp| + |\gamma_\varepsilon|\,|Fs - a_\varepsilon|\right) + \gamma_\varepsilon^2 \\
&\quad \leq 2|Fm|\left(\frac{1}{2\varepsilon|Fm|}\left|Fm - \gamma_\varepsilon Fs - a_\varepsilon^\perp\right|^2 + \frac{\varepsilon|Fm|}{2}(1+\gamma_\varepsilon^2) + \frac{1}{2\varepsilon|Fm|}|Fs - a_\varepsilon|^2\right) + \gamma_\varepsilon^2 \\
&\quad = \frac{1}{\varepsilon}|Fm - \gamma_\varepsilon Fs - a_\varepsilon^\perp|^2 + \frac{1}{\varepsilon}|Fs - a_\varepsilon|^2 + \gamma_\varepsilon^2 + \varepsilon|Fm|^2\left(\gamma_\varepsilon^2 + 1\right) \\
&\quad = W_\varepsilon(F) + \varepsilon|Fm|^2\left(\gamma_\varepsilon^2 + 1\right) \;\leq\; (\varepsilon|Fm|^2 + 1)W_\varepsilon(F) + \varepsilon|Fm|^2,
\end{aligned}
$$

which yields (7.19).
In order to verify (7.20) we calculate using Young's inequality,

$$
\begin{aligned}
|Fm| &\leq |Fm - \gamma_\varepsilon Fs - a_\varepsilon^\perp| + |a_\varepsilon^\perp + \gamma_\varepsilon Fs| \\
&\leq \frac{1}{2\sqrt{\varepsilon}}|Fm - \gamma_\varepsilon Fs - a_\varepsilon^\perp|^2 + |\gamma_\varepsilon|\,|Fs - a_\varepsilon| + \frac{\sqrt{\varepsilon}}{2} + |\gamma_\varepsilon a_\varepsilon + a_\varepsilon^\perp| \\
&\leq \frac{\sqrt{\varepsilon}}{2}W_\varepsilon(F) + \frac{\sqrt{\varepsilon}}{2} + |\gamma_\varepsilon| + 1 \;\leq\; \sqrt{\varepsilon}W_\varepsilon(F) + \frac{\sqrt{\varepsilon}}{2} + \frac{1}{2\sqrt{\varepsilon}} + 1.
\end{aligned}
$$

Since the condition $|Fm|^2\varepsilon \geq (2+\varepsilon)^2$ implies $\frac{1}{2}|Fm| \geq \frac{1}{2\sqrt{\varepsilon}} + 1$, we finally have $\frac{1}{2}|Fm| \leq \sqrt{\varepsilon}W_\varepsilon(F) + \frac{\sqrt{\varepsilon}}{2}$. □

*PROOF of Theorem 7.12:* By Lemma 7.14 we get that for all $F \in \mathbb{R}^{2\times 2}$ that

$$
W_\varepsilon(F) \geq f_\varepsilon(|Fm|), \tag{7.21}
$$

where

$$
f_\varepsilon : \; \mathbb{R}_0^+ \to \mathbb{R}, \qquad z \mapsto \begin{cases} h_\varepsilon(z), & \text{if } z^2\varepsilon \leq (2+\varepsilon)^2, \\ \frac{z}{2\sqrt{\varepsilon}} - b_\varepsilon, & \text{otherwise} \end{cases}
$$

with $b_\varepsilon \geq 1/2$ such that $f_\varepsilon$ is continuous. This lower bound $f_\varepsilon$, however, is neither convex nor polyconvex. Therefore a better bound is needed. Observe that $h_\varepsilon$ is convex if $z^2\varepsilon \leq 1/3$ and concave else. By $\lim_{z\to 0} h_\varepsilon'(z) = 0$ there exists $\alpha \in (0, 1/3)$ such that the function

$$
g_\varepsilon : \; \mathbb{R}_0^+ \to \mathbb{R}, \qquad z \mapsto \begin{cases} h_\varepsilon(z), & \text{if } z^2\varepsilon \leq \alpha, \\ h_\varepsilon'(\sqrt{\frac{\alpha}{\varepsilon}})z + d_\varepsilon, & \text{otherwise,} \end{cases}
$$

fulfills $g_\varepsilon(z) \leq f_\varepsilon(z)$ for all $z \in \mathbb{R}_0^+$ and all $\varepsilon > 0$. Here $d_\varepsilon$ is chosen in such a way that $g_\varepsilon$ is continuous. Note that $g_\varepsilon$ is monotone and convex by construction. Hence, in view of (7.21)

$$W_\varepsilon^{\mathrm{qc}}(F) \geq g_\varepsilon(|Fm|) \tag{7.22}$$

for all $F \in \mathbb{R}^{2\times 2}$ and $\varepsilon > 0$. This yields $W_\varepsilon^{\mathrm{qc}}(F) \geq h_\varepsilon(|Fm|)$, if $\varepsilon$ is sufficiently small. As a consequence we obtain $\lim_{\varepsilon\to 0} W_\varepsilon^{\mathrm{qc}}(F) \geq |Fm|^2 - 1$. Together with the upper bound $W_\varepsilon^{\mathrm{qc}} \leq W^{\mathrm{qc}}$ for all $\varepsilon > 0$ this provides the claim for all $F \in \mathcal{N}^{(2)}$. For the remaining $F \in \mathbb{R}^{2\times 2} \setminus \mathcal{N}^{(2)}$ the assertion follows immediately from Corollary 7.7. □

### Regularity of $W_\varepsilon^{\mathrm{qc}}$

It is by now a well-known fact that (finite-valued) quasiconvex functions are locally Lipschitz continuous, see i.e. [25, Section 4.1.1, Theorem 1.1 (iv)] or Lemma 5.2. The next theorem, where we summarize our findings concerning the regularity of $W_{\varepsilon;2,2}^{\mathrm{qc}}$ shows even more.

**Theorem 7.15** *For $\varepsilon > 0$ the relaxed energy density $W_\varepsilon^{\mathrm{qc}}$ is a $C^1$-function. Furthermore it holds $W_\varepsilon^{\mathrm{qc}} \in C_{loc}^{1,1}\left(\mathbb{R}^{2\times 2}\right)$, precisely*

$$\left|\nabla W_\varepsilon^{\mathrm{qc}}(F+X) - \nabla W_\varepsilon^{\mathrm{qc}}(F)\right| \leq \frac{c}{\varepsilon}\max\left\{1, W_\varepsilon^{\mathrm{qc}}(F)\right\}|X| \tag{7.23}$$

*for all $X, F \in \mathbb{R}^{2\times 2}$ with the constant $c$ independent of $\varepsilon, X$ and $F$.*

**Remark 7.16** *1. Notice that $W_\varepsilon$ as in* (7.1) *is finite-valued by Lemma 7.1. This property is essential for the subsequent proof.*
*2. The proof of Theorem 7.15 mainly follows the lines of [10, Theorem 3.1]. However, $W_\varepsilon$ lacks the growth conditions required for an analogue procedure via representation of $W_\varepsilon^{\mathrm{qc}}$ by probability measures as stated in* (3.11)*. In particular, it is impossible to construct a sub-probability measure $\nu$ in the sense of [10, Proposition 3.6], so that the widely used formula*

$$\nabla W_\varepsilon^{\mathrm{qc}}(\bar{\nu}) = \int \nabla W_\varepsilon \, \mathrm{d}\nu,$$

*where $\bar{\nu}$ is the center of mass, cannot be used in the present context. To overcome this difficulty we work with a more direct approach by applying Proposition 3.12 and exploiting the special structure of $W_\varepsilon$ and its relaxation to gain explicit control on the associated difference quotients.*

*PROOF.* First let us show that $W_\varepsilon^{\mathrm{qc}}$ is differentiable. Then in view of its separate convexity the continuity of $\nabla W_\varepsilon^{\mathrm{qc}}$ follows immediately from Lemma 5.3. Our goal for

the moment is the construction of a suitable upper bound on the difference quotients of $W_\varepsilon^{\mathrm{qc}}$, which will help later to prove upper semi-differentiability.
Let $F \in \mathbb{R}^{2\times 2}$. With the notation of (7.4) we can write

$$W_\varepsilon(F) = \frac{1}{\varepsilon}|F(\mathbb{I} - \gamma_\varepsilon(F)s \otimes m) - Q_\varepsilon(F)|^2 + \gamma_\varepsilon^2(F),$$

where $Q_\varepsilon(F) = a_\varepsilon(F) \otimes a_\varepsilon(F)^\perp \in \mathrm{SO}(2)$. For this proof we suppress the dependence of $\varepsilon$ in our notation and simply write $a_F = a_\varepsilon(F)$, $\gamma_F = \gamma_\varepsilon(F)$ and $Q_F = Q_\varepsilon(F)$. We calculate for $X \in \mathbb{R}^{2\times 2}$,

$$\begin{aligned} W_\varepsilon(F+X) &\le \frac{1}{\varepsilon}\big|(F+X)(\mathbb{I} - \gamma_F s \otimes m) - Q_F\big|^2 + \gamma_F^2 \\ &= W_\varepsilon(F) + \frac{2}{\varepsilon} X : \Big(\big(F(\mathbb{I} - \gamma_F s \otimes m) - Q_F\big)\,(\mathbb{I} - \gamma_F m \otimes s)\Big) + \frac{1}{\varepsilon}\big|X(\mathbb{I} - \gamma_F s \otimes m)\big|^2 \\ &\le W_\varepsilon(F) + \frac{2}{\varepsilon}\,(Y_F : X) + \frac{1}{\varepsilon}(2+\gamma_F^2)|X|^2 \end{aligned} \tag{7.24}$$

with the definition $Y_F = \big(F(\mathbb{I} - \gamma_F s \otimes m) - Q_F\big)\,(\mathbb{I} - \gamma_F m \otimes s)$. In order to achieve similar estimates also for the quasiconvex envelpe of $W_\varepsilon$ we use the formula

$$W_\varepsilon^{\mathrm{qc}}(F) = \inf_{\varphi \in W_0^{1,\infty}((0,1)^2;\mathbb{R}^2)} \int_{(0,1)^2} W_\varepsilon(F + \nabla\varphi)\,\mathrm{d}x \tag{7.25}$$

resulting from Proposition 3.12. Let $\{\varphi^k\}_{k\in\mathbb{N}}$ be a minimizing sequence for the right-hand side of (7.25). Then (7.24) yields for all $k \in \mathbb{N}$ sufficiently large,

$$\begin{aligned} W_\varepsilon^{\mathrm{qc}}(F+X) &\le \int_{(0,1)^2} W_\varepsilon(F + \nabla\varphi^k + X)\,\mathrm{d}x \\ &\le \int_{(0,1)^2} W_\varepsilon(F+\nabla\varphi^k) + \frac{2}{\varepsilon}\big(Y_{(F+\nabla\varphi^k)} : X\big) + \frac{1}{\varepsilon}\big(2 + W_\varepsilon(F+\nabla\varphi^k)\big)|X|^2\,\mathrm{d}x. \\ &\le \int_{(0,1)^2} W_\varepsilon(F+\nabla\varphi^k)\,\mathrm{d}x + \frac{2}{\varepsilon}(A^k : X) + \frac{1}{\varepsilon}\big(3 + W_\varepsilon^{\mathrm{qc}}(F)\big)|X|^2. \end{aligned} \tag{7.26}$$

In the last inequality the estimate $\int_{(0,1)^2} W_\varepsilon(F+\nabla\varphi^k)\,\mathrm{d}x \le W_\varepsilon^{\mathrm{qc}}(F) + 1$ for $k$ large enough was applied, as well as the definition $A^k = \int_{(0,1)^2} Y_{(F+\nabla\varphi^k)}\,\mathrm{d}x$ for $k \in \mathbb{N}$. For simplicity we omit the dependence of $F$ in the notation of $A^k$. Next the limit $k \to \infty$ in (7.26) has to be analyzed. For this purpose we show

$$\begin{aligned} |A^k| &\le \int_{(0,1)^2} |Y_{(F+\nabla\varphi^k)}|\,\mathrm{d}x \le \int_{(0,1)^2} \sqrt{2 + \gamma^2_{(F+\nabla\varphi^k)}}\sqrt{\varepsilon W_\varepsilon(F+\nabla\varphi^k)}\,\mathrm{d}x \\ &\le \int_{(0,1)^2} \sqrt{\varepsilon}\big(1 + W_\varepsilon(F+\nabla\varphi^k)\big)\,\mathrm{d}x \le \sqrt{\varepsilon}(2 + W_\varepsilon^{\mathrm{qc}}(F)) \end{aligned} \tag{7.27}$$

for $k \in \mathbb{N}$ sufficiently large. Hence $\{A^k\}_{k\in\mathbb{N}}$ is bounded and after restriction to a suitable subsequence it holds $A^k \to A$ as $k \to \infty$ for some $A \in \mathbb{R}^{2\times 2}$. After taking the limit (7.26) turns into

$$W_\varepsilon^{\mathrm{qc}}(F+X) \quad \leq \quad W_\varepsilon^{\mathrm{qc}}(F) + \frac{2}{\varepsilon}(A:X) + \frac{1}{\varepsilon}\big(3 + W_\varepsilon^{\mathrm{qc}}(F)\big)|X|^2. \tag{7.28}$$

Note that $A$ may depend on the choice of the minimizing sequence $\{\varphi^k\}_{k\in\mathbb{N}}$. In order to render inequality (7.28) independent of the latter we define the function $V : \mathbb{R}^{2\times 2} \to \mathbb{R}$, $V(X) = \frac{2}{\varepsilon}\inf_{A\in\mathcal{A}}(A:X)$ with

$$\mathcal{A} = \Big\{ A \in \mathbb{R}^{2\times 2} \mid A \text{ is accumulation point of } \{A^k\}_{k\in\mathbb{N}},\ \text{where} \\ A^k = \textstyle\fint_{(0,1)^2} Y_{(F+\nabla\varphi^k)}\,\mathrm{d}x \text{ and } \{\varphi^k\}_{k\in\mathbb{N}} \text{ a minimizing sequence of (7.25)} \Big\}.$$

Then,

$$W_\varepsilon^{\mathrm{qc}}(F+X) \quad \leq \quad W_\varepsilon^{\mathrm{qc}}(F) + V(X) + \frac{1}{\varepsilon}\big(3 + W_\varepsilon^{\mathrm{qc}}(F)\big)|X|^2. \tag{7.29}$$

Due to (7.27) we know that the set $\mathcal{A}$ is bounded. Besides, $\mathcal{A}$ is closed. This can be seen by considering a sequence $\{A_l\}_{l\in\mathbb{N}} \subset \mathcal{A}$ with $\lim_{l\to\infty} A_l = A$ and proving $A \in \mathcal{A}$. By definition there is a sequence $\{A_l^k\}_{k\in\mathbb{N}}$ for each $l \in \mathbb{N}$ such that $A_l^k = \fint_{(0,1)^2} Y_{(F+\nabla\varphi_l^k)}\,\mathrm{d}y$ with $\{\varphi_l^k\}_{k\in\mathbb{N}}$ being minimal for (7.25) and $\lim_{k\to\infty} A_l^k = A_l$. Then a diagonal sequence $\{A_l^{k_l}\}_{l\in\mathbb{N}}$ can be extracted such that $\lim_{l\to\infty} A_l^{k_l} = A$ and $\lim_{l\to\infty} \fint_{(0,1)^2} W_\varepsilon(F + \nabla\varphi_l^{k_l})\,\mathrm{d}x = W_\varepsilon^{\mathrm{qc}}(F)$. Thus, $A \in \mathcal{A}$. Indeed, it has been shown that $\mathcal{A}$ is compact in $\mathbb{R}^{2\times 2}$, so that the infimum in the definition of $V$ is attained, i.e. $V(X) = \frac{2}{\varepsilon}\min_{A\in\mathcal{A}}(A:X)$, and $V$ is upper semidifferentiable regarding Lemma 5.6.
Eventually, by (7.29) we can find an upper bound for the difference quotient of $W_\varepsilon^{\mathrm{qc}}$, that is

$$\Lambda_t(X) := \frac{W_\varepsilon^{\mathrm{qc}}(F+tX) - W_\varepsilon^{\mathrm{qc}}(F)}{t} \leq V(X) + \frac{1}{\varepsilon}t\,\big(3 + W_\varepsilon^{\mathrm{qc}}(F)\big)|X|^2$$

for $t > 0$. After a limiting process

$$\Lambda(X) := \limsup_{t\to 0^+} \Lambda_t(X) \leq V(X).$$

Being limes superior of the separately convex functions $\Lambda_t$, $\Lambda$ is separately convex itself. So far we have found out the following, $V$ is upper semidifferentiable, $\Lambda$ is separately convex, $\Lambda \leq V$ on $\mathbb{R}^{2\times 2}$ and $\Lambda(0) = 0 = V(0)$. It is immediate from Corollary 5.7 that $\Lambda$ and $V$ are differentiable at 0 with $\nabla\Lambda(0) = \nabla V(0)$.
Since $\Lambda$ is positively homogenous of degree one, it can be represented by the formula

$$\Lambda(X) = \nabla\Lambda(0) : X = \nabla V(0) : X \tag{7.30}$$

for all $X \in \mathbb{R}^{2\times 2}$. Our next claim is that $W_\varepsilon^{\mathrm{qc}}$ is upper semidifferentiable in $F$. Let us argue via contradiction. If this assertion is false, there exist $\delta > 0$ and a sequence $\{X^j\}_{j\in\mathbb{N}} \subset \mathbb{R}^{2\times 2}$ with $X^j \to 0$ as $j \to \infty$ such that

$$\frac{W_\varepsilon^{\mathrm{qc}}(F + X^j) - W_\varepsilon^{\mathrm{qc}}(F) - \Lambda(X^j)}{|X^j|} > \delta.$$

We define $\{\hat{X}^j\}_{j\in\mathbb{N}}$ by $\hat{X}^j = \frac{X^j}{|X^j|}$ for $j \in \mathbb{N}$, so that (possibly after passing to a subsequence) $\hat{X}^j \to \hat{X}_\infty \in \mathbb{R}^{2\times 2}$ as $j \to \infty$ and we infer from Lemma 5.2 that $W_\varepsilon^{\mathrm{qc}}$ is locally Lipschitz continuous. Together with the positive one-homogeneity of $\Lambda$ and (7.30) we obtain for $j \in \mathbb{N}$ large enough,

$$\begin{aligned}\delta &< \frac{\left|W_\varepsilon^{\mathrm{qc}}(F + X^j) - W_\varepsilon^{\mathrm{qc}}(F + |X^j|\hat{X}_\infty)\right| + W_\varepsilon^{\mathrm{qc}}(F + |X^j|\hat{X}_\infty) - W_\varepsilon^{\mathrm{qc}}(F)}{|X^j|} \\ &\qquad\qquad\qquad\qquad - \frac{\Lambda(X^j) - \Lambda(|X^j|\hat{X}_\infty)}{|X^j|} - \Lambda(\hat{X}_\infty) \\ &\leq L\,|\hat{X}^j - \hat{X}_\infty| + \frac{W_\varepsilon^{\mathrm{qc}}(F + |X^j|\hat{X}_\infty) - W_\varepsilon^{\mathrm{qc}}(F)}{|X^j|} + \Lambda(\hat{X}_\infty - \hat{X}^j) - \Lambda(\hat{X}_\infty),\end{aligned}$$

where $L = \mathrm{lip}(W_\varepsilon^{\mathrm{qc}}; B(F, 1))$. Thus as $j \to \infty$,

$$\delta \leq \limsup_{j\to\infty} \frac{W_\varepsilon^{\mathrm{qc}}(F + |X^j|\hat{X}_\infty) - W_\varepsilon^{\mathrm{qc}}(F)}{|X^j|} - \Lambda(\hat{X}_\infty) = \Lambda(\hat{X}_\infty) - \Lambda(\hat{X}_\infty) = 0.$$

Obviously, this is a contradiction to the initial choice of $\delta$.

All in all we have shown so far that $W_\varepsilon^{\mathrm{qc}}$ is differentiable in $F$, which results from its separate convexity and its lower semidifferentiability in combination with Corollary 5.7 applied for $f = g = W_\varepsilon^{\mathrm{qc}}$. Notice that (7.30) additionally gives a characterization of the gradient of $W_\varepsilon^{\mathrm{qc}}$, namely $\nabla W_\varepsilon^{\mathrm{qc}}(F) = \nabla V(0)$. Finally the arbitrariness of $F$ immediately yields the first part of Theorem 7.15.

Our next aim is to prove $W_\varepsilon^{\mathrm{qc}} \in C^{1,1}_{loc}(\mathbb{R}^{2\times 2})$. To this end we take $F \in \mathbb{R}^{2\times 2}$ and define the separately convex auxiliary function

$$h : \mathbb{R}^{2\times 2} \to \mathbb{R}, \qquad h(X) = W_\varepsilon^{\mathrm{qc}}(F + X) - W_\varepsilon^{\mathrm{qc}}(F) - V(X),$$

for which we infer from (7.29) that

$$h(X) \leq \frac{1}{\varepsilon}\big(3 + W_\varepsilon^{\mathrm{qc}}(F)\big)\,|X|^2 \tag{7.31}$$

for all $X \in \mathbb{R}^{2\times 2}$. Here again we dispense with marking the dependence on $F$ in our notation of $h$. In order to derive a similar upper bound for $|h|$ one computes

$$h(X) \geq \inf_{|Y|<|X|} h(Y) \geq (1 - 2^4) \sup_{|Y|<|X|} h(Y) = \frac{(1 - 2^4)}{\varepsilon}(3 + W_\varepsilon^{\mathrm{qc}}(F))|X|^2,$$

where Lemma 5.4 and (7.31) were used for the second and third inequality, repsectively. In combination with (7.31) this implies $|h(X)| \leq \frac{(2^4-1)}{\varepsilon}\big(3 + W_\varepsilon^{\mathrm{qc}}(F)\big)\ |X|^2$ for all $X \in \mathbb{R}^{2\times 2}$. Moreover, one infers from Lemma 5.2 that $h$ is locally Lipschitz continuous and that

$$\begin{aligned} |\nabla h(X)| &\leq \operatorname{lip}\big(h; B(0,2|X|)\big) \leq \frac{\operatorname{osc}\big(h; B(0,4|X|)\big)}{|X|} \leq 2\,\frac{\sup\{|h(Y)| : Y \in B(0,4|X|)\}}{|X|} \\ &\leq \frac{2^5(2^4-1)}{\varepsilon}\big(3 + W_\varepsilon^{\mathrm{qc}}(F)\big)|X| \leq \frac{3\cdot 2^6(2^4-1)}{\varepsilon}\max\{1, W_\varepsilon^{\mathrm{qc}}(F)\}|X|. \quad (7.32) \end{aligned}$$

Since $\nabla V(X) = \nabla V(0) = \nabla W_\varepsilon^{\mathrm{qc}}(F)$ for all $X \in \mathbb{R}^{2\times 2}$, it holds $\nabla h(X) = \nabla W_\varepsilon^{\mathrm{qc}}(F+X) - \nabla W_\varepsilon^{\mathrm{qc}}(F)$, which in combination with (7.32) yields the postulated estimate. $\square$

### Convergence of the gradients of $W_\varepsilon^{\mathrm{qc}}$

This section is concerned with the gradients of $W_\varepsilon^{\mathrm{qc}}$ and their asymptotic behavior as $\varepsilon$ tends to zero. One may expect that the corresponding limit is related to the gradient of the effective elastically rigid energy density $W^{\mathrm{qc}}$. Speaking of differentiability in connection with $W^{\mathrm{qc}}$, however, is not possible in the usual sense. Indeed, $W^{\mathrm{qc}}$ is infinite on $\mathbb{R}^{2\times 2} \setminus \overline{\mathfrak{S}}$, where $\mathfrak{S} = \{F \in \mathbb{R}^{2\times 2} \mid \det F = 1, |Fs| < 1\}$ is an open subset of a smooth hypersurface in $\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$. In order to find a reasonable notion for the gradient of $W^{\mathrm{qc}}$, we use for fixed $F \in \mathfrak{S}$ the unique additive decomposition of $G \in \mathbb{R}^{2\times 2}$ into a tangential and a normal part with respect to $\mathfrak{S}$ in $F$. So we write $G = G^{\tan} + G^{\mathrm{norm}}$, where $G^{\tan} \in T_F(\mathfrak{S})$ and $G^{\mathrm{norm}}$ lies in the associated normal space. Here $T_F(\mathfrak{S})$ stands for the three-dimensional tangent space of $\mathfrak{S}$ in $F$, which is given by

$$\begin{aligned} T_F(\mathfrak{S}) = \big\{Y \in \mathbb{R}^{2\times 2} \mid{} & \text{there exist } \eta > 0 \text{ and} \\ & \text{a } C^\infty\text{-smooth curve } k : [-\eta,\eta] \to \mathfrak{S} \text{ with } k'(0) = Y \text{ and } k(0) = F\big\}. \end{aligned}$$

Now let us set

$$\widetilde{W}(F) = |Fm|^2 - 1 \qquad \text{for all } F \in \mathbb{R}^{2\times 2}. \tag{7.33}$$

Then $\widetilde{W}$ is finite, $C^\infty$-smooth and it coincides with $W^{\mathrm{qc}}$ on $\mathcal{N}^{(2)}$. Thus, by the gradient of $W^{\mathrm{qc}}$ in $F \in \mathfrak{S}$ we refer to the projection of $\nabla\widetilde{W}(F)$ on $T_F(\mathfrak{S})$ from now on. With these considerations in mind the announced result reads as follows.

**Theorem 7.17** *Let $\varepsilon > 0$ and $\widetilde{W} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be as defined in (7.33). If $F \in \mathfrak{S}$, $r > 0$ and $\delta \in (0,1)$ satisfy $|W_\varepsilon^{\mathrm{qc}}(G) - W^{\mathrm{qc}}(G)| \leq \delta$ for all $G \in \mathfrak{S} \cap B(F,r)$, then there exists a constant $c = c(F,r) > 0$ such that*

$$\Big|\big(\nabla W_\varepsilon^{\mathrm{qc}}(F) - \nabla\widetilde{W}(F)\big)^{\tan}\Big| \leq c\,\sqrt{\delta}.$$

*PROOF.* The idea of the proof is to apply the one-dimensional result of Proposition 5.8 to $W_\varepsilon^{\mathrm{qc}}$ and $W^{\mathrm{qc}}$ on selected rank-one lines. Let $F \in \mathfrak{S}$, $r > 0$ and $\delta \in (0,1)$ be as in the statement of the theorem. We begin by constructing appropriate rank-one lines lying in $\mathfrak{S}$ and passing through $F$. This will automatically provide an explicit representation for the three-dimensional tangent space $T_F(\mathfrak{S})$. In fact, there is $\eta \in (0,1)$ depending on $r$ and $F$ such that the curves

$$\begin{aligned} k^j : [-\eta, \eta] &\rightarrow \mathfrak{S} \cap B(F,r) \\ t &\mapsto F + tFa^j \otimes b^j, \quad j \in \{1,2,3\} \end{aligned} \tag{7.34}$$

with $a^1 = s$, $b^1 = m$, $a^2 = m$, $b^2 = s$, $a^3 = s+m$ and $b^3 = s-m$ are well-defined. Notice that $k^j$ actually maps $[-\eta,\eta]$ into $\mathfrak{S}\cap B(F,r)$, since $|tFa^j\otimes b^j| < r$ and $|Fs+tFa^j\otimes b^j s| \leq |Fs| + |t|\,|Fa^j \otimes b^j| < 1$ for $j \in \{1,2,3\}$ and $|t| < \eta$, if $\eta$ is chosen sufficiently small. Besides, it holds $\det(F + tFa^j \otimes b^j) = \det F = 1$, since $a^j \perp b^j$ for $j \in \{1,2,3\}$.
As a consequence of (7.34) we observe

$$T_F(\mathfrak{S}) \supset \operatorname{span}\big\{(k^j)'(0)\,|\,j\in\{1,2,3\}\big\} = \operatorname{span}\big\{Fa^j\otimes b^j\,|\,j\in\{1,2,3\}\big\}. \tag{7.35}$$

Due to the linear independence of $Fa^j \otimes b^j$, $j \in \{1,2,3\}$ we even have equality in (7.35), meaning

$$\begin{aligned} T_F(\mathfrak{S}) &= \operatorname{span}\big\{Fa^j\otimes b^j\,|\,j\in\{1,2,3\}\big\} \\ &= \operatorname{span}\,\{Fs\otimes m,\ Fm\otimes s,\ F(s+m)\otimes(s-m)\}\,. \end{aligned} \tag{7.36}$$

Next we set $\mu = \eta\sqrt{\delta}$ and we define the maps

$$\begin{aligned} w_\varepsilon^j : [-\mu,\mu] &\to \mathbb{R} \\ t &\mapsto W_\varepsilon^{\mathrm{qc}}(k^j(t)) \end{aligned} \qquad \text{and} \qquad \begin{aligned} v^j : [-\mu,\mu] &\to \mathbb{R} \\ t &\mapsto \widetilde{W}(k^j(t)) \end{aligned}$$

with $j \in \{1,2,3\}$. Notice that $w_\varepsilon^j$ are convex functions, since the quasiconvex $W_\varepsilon^{\mathrm{qc}}$ is convex along the rank-one lines $k^j$. According to Theorem 7.15 one has $w_\varepsilon^j \in C^1([-\mu,\mu])$. Moreover, it is obvious that $v^j \in C^2([-\mu,\mu])$ as a composition of smooth functions. By assumption and (7.34) it holds

$$|w_\varepsilon^j(t) - v^j(t)| \leq \delta$$

for all $t \in [-\mu,\mu]$ and $j \in \{1,2,3\}$. Hence, $v^j$ and $w_\varepsilon^j$ fulfill all the requirements of Proposition 5.8. In view of the characterization (7.36) of the tangent space $T_F(\mathfrak{S})$ Proposition

5.8 yields

$$
\begin{aligned}
\left|\left(\nabla W_\varepsilon^{\mathrm{qc}}(F)-\nabla\widetilde{W}(F)\right)^{\tan}\right| \;&\le c(F)\;\sum_{j=1}^{3}\left|\left(\nabla W_\varepsilon^{\mathrm{qc}}(F)-\nabla\widetilde{W}(F)\right):(Fa^j\otimes b^j)\right| \\
=\quad c(F)\;\sum_{j=1}^{3}\left|\,(w_\varepsilon^j)'(0)-\;(v^j)'(0)\right| \;&\le\; c(F)\Big(\frac{6\delta}{\mu}+\mu\sum_{j=1}^{3}\max_{t\in[-\mu,\mu]}|(v^j)''(t)|\Big) \\
=c(F)\Big(\frac{6\sqrt{\delta}}{\eta}+2\sqrt{\delta}\eta\sum_{j=1}^{3}\left|(Fa^j\otimes b^j)m\right|^2\Big) &\le c(F,r)\sqrt{\delta}.
\end{aligned}
$$

This shows the assertion of Theorem 7.17. ☐

### 7.3.2. A $\Gamma$-convergence result with elastically rigid limit

One of the main achievements of this thesis is the $\Gamma$-convergence result which has already been stated in the introduction as Theorem 1.2. Here we also point to [20] as a reference. Recall that we are still working within the framework of a single-slip model with linear hardening and quadratic elastic energy ($p=q=2$), so that the condensed energy density is given by $W_\varepsilon = W_{\varepsilon;2,2}$ and $W=W_2$ is its elastically rigid counterpart.

#### Formulation of the main theorem

First let us give an exact and detailed formulation of the theorem.

**Theorem 7.18** *Suppose $\Omega\subset\mathbb{R}^2$ is a bounded Lipschitz domain and $p=q=2$. For $\varepsilon>0$ let $E_\varepsilon$ and $E$ be the functionals defined by (7.7) and (7.8). Then, $E_\varepsilon$ converges to $E$ in the sense of $\Gamma$-convergence with respect to the strong $L^1$-topology as $\varepsilon$ tends to zero. In symbols,*

$$E_\varepsilon \xrightarrow{\Gamma} E \quad \text{as} \;\; \varepsilon\to 0.$$

*This goes along with a compactness result for sequences of bounded energy. More precisely, the following assertions are satisfied:*

Compactness and lower bound inequality: *If $\{u_\varepsilon\}_{\varepsilon>0}\subset W^{1,1}(\Omega;\mathbb{R}^2)$ is a sequence of bounded energy, i.e. for all $\varepsilon>0$ it holds $E_\varepsilon[u_\varepsilon]=\int_\Omega W_\varepsilon(\nabla u_\varepsilon)\,\mathrm{d}x\le B<\infty$, and if $(u_\varepsilon)_\Omega=0$ for all $\varepsilon>0$, there exist a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and a function $u\in W^{1,2}(\Omega;\mathbb{R}^2)$ such that $u_{\varepsilon_k}\to u$ in $L^1(\Omega;\mathbb{R}^2)$ as $k\to\infty$ with $u_\Omega=0$ and $\nabla u\in\mathcal{N}^{(2)}$ almost everywhere in $\Omega$. Furthermore, one has the lower bound inequality*

$$\liminf_{k\to\infty}E_{\varepsilon_k}[u_{\varepsilon_k}]=\liminf_{k\to\infty}\int_\Omega W_{\varepsilon_k}(\nabla u_{\varepsilon_k})\,\mathrm{d}x\ge\int_\Omega W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x=E[u]. \tag{7.37}$$

Upper bound inequality: *For every* $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ *with* $u_\Omega = 0$ *and* $\nabla u \in \mathcal{N}^{(2)}$ *almost everywhere in* $\Omega$ *there is a recovery sequence* $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}} \subset W^{1,1}(\Omega;\mathbb{R}^2)$ *with* $(u_{\varepsilon_k})_\Omega = 0$ *for all* $k \in \mathbb{N}$ *such that* $u_{\varepsilon_k} \to u$ *in* $L^1(\Omega;\mathbb{R}^2)$ *as* $k \to \infty$ *and*

$$\limsup_{k\to\infty} E_{\varepsilon_k}[u_{\varepsilon_k}] = \limsup_{k\to\infty} \int_\Omega W_{\varepsilon_k}(\nabla u_{\varepsilon_k})\,\mathrm{d}x \le \int_\Omega W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x = E[u]. \tag{7.38}$$

In the following we give a proof of Theorem 7.18, which we subdivide into the natural three steps. These are compactness and the lower and upper bound.

### Compactness

The key to the proof of compactness for Theorem 7.18 is the special structure of bounded energy sequences $\{u_\varepsilon\}_{\varepsilon>0}$ of $E_\varepsilon$. As Lemma 7.19 and the subsequent corollary will show, the gradients of functions belonging to bounded energy sequences fall into a quadratically integrable part and a term converging to zero in $L^1$ as $\varepsilon$ tends to zero. To make things mathematically exact let us recall the additive decomposition for $u_\varepsilon \in W^{1,1}(\Omega;\mathbb{R}^2)$ with $\varepsilon > 0$ from (7.10). It reads

$$\nabla u_\varepsilon = A_\varepsilon + H_\varepsilon, \tag{7.39}$$

where $A_\varepsilon = a_\varepsilon \otimes s + g_\varepsilon \otimes m$ and $H_\varepsilon = h_\varepsilon \otimes s + f_\varepsilon \otimes m$. Concerning the definition of $a_\varepsilon$, $h_\varepsilon$, $g_\varepsilon$ and $f_\varepsilon$ we refer to Section 7.1.3, especially to (7.9).

**Lemma 7.19** *Suppose* $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,1}(\Omega;\mathbb{R}^2)$ *is a sequence which satisfies* $E_\varepsilon[u_\varepsilon] < B$ *for all* $\varepsilon > 0$. *Then, using the notation of Section 7.1.3 with* $p = q = 2$, *it holds*

*(i)* $\|a_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^2)} \le 1$, *(ii)* $\|g_\varepsilon\|_{L^2(\Omega;\mathbb{R}^2)} \le (1+\sqrt{\varepsilon})\sqrt{B+|\Omega|}$,

*(iii)* $\|h_\varepsilon\|_{L^2(\Omega;\mathbb{R}^2)} \le \sqrt{\varepsilon B}$, *(iv)* $\|f_\varepsilon\|_{L^1(\Omega;\mathbb{R}^2)} \le \sqrt{\varepsilon}B$.

*Further,* $\{A_\varepsilon\}_{\varepsilon>0}$ *is uniformly bounded in* $L^2(\Omega;\mathbb{R}^{2\times 2})$ *and* $\{H_\varepsilon\}_{\varepsilon>0}$ *is uniformly bounded in* $L^1(\Omega;\mathbb{R}^{2\times 2})$ *and equi-integrable.*

*PROOF.* Since $\{a_\varepsilon\}_{\varepsilon>0} \subset \mathbb{S}^1$ by definition, (i) is immediate. For the verification of (ii) we estimate

$$\begin{aligned} \|g_\varepsilon\|_{L^2(\Omega;\mathbb{R}^2)} &= \|(\nabla u_\varepsilon)m - \gamma_\varepsilon(\nabla u_\varepsilon)s + \gamma_\varepsilon a_\varepsilon\|_{L^2(\Omega;\mathbb{R}^2)} \\ &\le \|(\nabla u_\varepsilon)m - \gamma_\varepsilon(\nabla u_\varepsilon)s - a_\varepsilon^\perp\|_{L^2(\Omega;\mathbb{R}^2)} + \|a_\varepsilon^\perp + \gamma_\varepsilon a_\varepsilon\|_{L^2(\Omega;\mathbb{R}^2)} \\ &\le \sqrt{\varepsilon B} + \Big(\textstyle\int_\Omega (1+\gamma_\varepsilon^2)\,\mathrm{d}x\Big)^{1/2} \le \sqrt{\varepsilon B} + \sqrt{B+|\Omega|} \\ &\le (1+\sqrt{\varepsilon})\sqrt{B+|\Omega|}\,, \end{aligned} \tag{7.40}$$

where (7.4) was used in the third and fourth inequality. Moreover, (7.4) directly implies the a priori estimates (iii) and $\|\gamma_\varepsilon\|_{L^2(\Omega)} \le \sqrt{B}$. Considering the definition $f_\varepsilon = \gamma_\varepsilon h_\varepsilon$ for all $\varepsilon > 0$ we obtain (iv) by Hölder's inequality. $\square$

The next corollary is an immediate consequence of Lemma 7.19.

**Corollary 7.20** *Under the assumptions of Lemma 7.19 there exist a subsequence* $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ *and functions* $a \in L^\infty(\Omega;\mathbb{R}^2)$ *and* $g \in L^2(\Omega;\mathbb{R}^2)$ *such that*

*(i)* $a_{\varepsilon_k} \overset{*}{\rightharpoonup} a$ *in* $L^\infty(\Omega;\mathbb{R}^2)$, *(ii)* $g_{\varepsilon_k} \rightharpoonup g$ *in* $L^2(\Omega;\mathbb{R}^2)$,

*(iii)* $h_{\varepsilon_k} \to 0$ *in* $L^2(\Omega;\mathbb{R}^2)$, *(iv)* $f_{\varepsilon_k} \to 0$ *in* $L^1(\Omega;\mathbb{R}^2)$.

*Further,*

$$A^k := A_{\varepsilon_k} \rightharpoonup A \text{ in } L^2(\Omega;\mathbb{R}^{2\times 2}) \quad \text{and} \quad H^k := H_{\varepsilon_k} \to 0 \text{ in } L^1(\Omega;\mathbb{R}^{2\times 2})$$

*for* $k \to \infty$, *where* $A := a \otimes s + g \otimes m \in L^2(\Omega;\mathbb{R}^{2\times 2})$.

*PROOF of Theorem 7.18 (Compactness).* In view of the decomposition (7.39) we infer from Lemma 7.19 that $\{\nabla u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^1(\Omega;\mathbb{R}^{2\times 2})$ and equi-integrable. Since $(u_\varepsilon)_\Omega = 0$ for all $\varepsilon > 0$ one gets by Poincaré's inequality that $\|u_\varepsilon\|_{W^{1,1}(\Omega;\mathbb{R}^2)} \leq C < \infty$. Then, by the weak compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ in $W^{1,1}(\Omega;\mathbb{R}^2)$, which is due to the equi-integrability of $\{\nabla u_\varepsilon\}_{\varepsilon>0}$, there exist a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and a function $u \in W^{1,1}(\Omega;\mathbb{R}^2)$ such that

$$u_{\varepsilon_k} \rightharpoonup u \quad \text{in} \quad W^{1,1}(\Omega;\mathbb{R}^2) \quad \text{as} \quad k \to \infty. \tag{7.41}$$

Then, the compactness of the embedding $W^{1,1}(\Omega;\mathbb{R}^2) \hookrightarrow L^1(\Omega;\mathbb{R}^2)$ immediately implies possibly after passing to another subsequence, again denoted $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$, that

$$u_{\varepsilon_k} \to u \quad \text{in} \quad L^1(\Omega;\mathbb{R}^2). \tag{7.42}$$

Hence, $u_\Omega = 0$. In the next step we compute explicitly the weak derivatives of $u$. From (7.41) and Corollary 7.20 one concludes using integration by parts that

$$\begin{aligned}\int_\Omega u_i\,\partial_j\varphi\,\mathrm{d}x &= \lim_{k\to\infty}\int_\Omega (u_{\varepsilon_k})_i\,\partial_j\varphi\,\mathrm{d}x = -\lim_{k\to\infty}\int_\Omega (\nabla u_{\varepsilon_k})_{ij}\,\varphi\,\mathrm{d}x \\ &= -\lim_{k\to\infty}\int_\Omega \big((A^k)_{ij} + (H^k)_{ij}\big)\varphi\,\mathrm{d}x = -\int_\Omega A_{ij}\varphi\,\mathrm{d}x\end{aligned} \tag{7.43}$$

for all $\varphi \in C_0^\infty(\Omega)$ and $i,j \in \{1,2\}$. Thus,

$$\nabla u = A \in L^2(\Omega;\mathbb{R}^{2\times 2}), \tag{7.44}$$

and indeed $u \in W^{1,2}(\Omega;\mathbb{R}^2)$.
It remains to prove $\nabla u \in \mathcal{N}^{(2)}$ almost everywhere, meaning $|(\nabla u)s| \leq 1$ and $\det \nabla u = 1$

almost everywhere in $\Omega$. The first of these properties results from Corollary 7.20 (i), (7.44) and the fact that $|a_{\varepsilon_k}| = 1$ almost everywhere in $\Omega$ for all $k \in \mathbb{N}$. At length,

$$\|(\nabla u)s\|_{L^\infty(\Omega;\mathbb{R}^2)} = \|As\|_{L^\infty(\Omega;\mathbb{R}^2)} = \|a\|_{L^\infty(\Omega;\mathbb{R}^2)} \leq \liminf_{k\to\infty} \|a_{\varepsilon_k}\|_{L^\infty(\Omega;\mathbb{R}^2)} = 1.$$

Regarding the recovery of the incompressibility constraint Corollary 6.24, which is a consequence of the generalized div-curl lemma Theorem 6.20, will turn out to be the crucial tool. First, we conclude from Lemma 7.4 that

$$\det \nabla u_{\varepsilon_k} \to 1 \quad \text{in } L^1(\Omega) \quad \text{for } \ k \to \infty. \tag{7.45}$$

This result, however, is not enough to guarantee $\det \nabla u = 1$ almost everywhere, for the mapping $v \mapsto \det \nabla v$ is not weakly continuous with respect to $W^{1,1}(\Omega;\mathbb{R}^2)$. In order to overcome this difficulty let us first rewrite $\det \nabla u_{\varepsilon_k}$ with $k \in \mathbb{N}$ in the form

$$\begin{aligned} \det \nabla u_{\varepsilon_k} &= \det\big((a_{\varepsilon_k} + h_{\varepsilon_k}) \otimes s + (g_{\varepsilon_k} + f_{\varepsilon_k}) \otimes m\big) \\ &= \det A^k - a_{\varepsilon_k} \cdot Jf_{\varepsilon_k} - h_{\varepsilon_k} \cdot Jg_{\varepsilon_k} + \det H^k, \end{aligned} \tag{7.46}$$

where $J$ is the counterclockwise rotation by $\pi/2$ in the plane. With $f_{\varepsilon_k}$ and $h_{\varepsilon_k}$ being parallel according to (7.9) one obtains $\det H^k = -h_{\varepsilon_k} \cdot Jf_{\varepsilon_k} = 0$. Since the second and third term in (7.46) converge to zero in $L^1(\Omega)$ in view of Corollary 7.20 (i)-(iv), we have

$$\det A^k - \det \nabla u_{\varepsilon_k} \to 0 \qquad \text{in } L^1(\Omega).$$

Then, $\det A^k \to 1$ in $L^1(\Omega)$ as $k \to \infty$ follows together with (7.45). Since $\det A^k = -a_{\varepsilon_k} \cdot Jg_{\varepsilon_k}$ for $k \in \mathbb{N}$, the sequence $\{\det A^k\}_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega)$ and therefore equi-integrable, so that Corollary 6.24 results in $\det A^k \rightharpoonup \det \nabla u$ in $L^1(\Omega)$. Summing up we find for all $\varphi \in C_0^\infty(\Omega)$ that

$$\int_\Omega \det \nabla u \, \varphi \,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega \det A^k \, \varphi \,\mathrm{d}x = \int_\Omega \varphi \,\mathrm{d}x.$$

Thus, $\det \nabla u = 1$ almost everywhere as asserted. □

### The lower bound

*PROOF of Theorem 7.18 (Lower bound inequality).* Suppose $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and $u$ are as in the statement of Theorem 7.18 and let $\delta > 0$. In view of the definition of limes inferior we can extract a subsequence $\{u_{\varepsilon_h}\}_{h\in\mathbb{N}}$ such that

$$E_{\varepsilon_h}[u_{\varepsilon_h}] \leq B_\delta := \liminf_{k\to\infty} E_{\varepsilon_k}[u_{\varepsilon_k}] + \delta.$$

As a consequence of Lemma 7.19 and (7.44) it holds, possibly after extracting a further subsequence, again denoted $\{u_{\varepsilon_h}\}_{h\in\mathbb{N}}$, that $g_{\varepsilon_h} \rightharpoonup g = (\nabla u)m$ in $L^2(\Omega;\mathbb{R}^2)$ as $h \to \infty$. Now we infer from (7.40) and the lower semicontinuity of the $L^2$-norm that

$$\begin{aligned}\|(\nabla u)m\|^2_{L^2(\Omega;\mathbb{R}^2)} &\leq \liminf_{h\to\infty} \|g_{\varepsilon_h}\|^2_{L^2(\Omega;\mathbb{R}^2)} \\ &\leq \liminf_{h\to\infty} (1+\sqrt{\varepsilon_h})^2(B_\delta + |\Omega|) = B_\delta + |\Omega|.\end{aligned}$$

So finally one obtains

$$\int_\Omega W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x = \int_\Omega |(\nabla u)m|^2 - 1\,\mathrm{d}x \quad \leq \quad B_\delta \;=\; \liminf_{k\to\infty} E_{\varepsilon_k}[u_{\varepsilon_k}] + \delta$$

and the proof is concluded by the arbitrariness of $\delta$. □

### The upper bound

In order to verify the upper bound inequality of Theorem 7.18 one needs to construct a recovery sequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ which realizes the passage of $E_{\varepsilon_k}$ to the rigid limiting functional $E$ as $k$ tends to infinity. This imposes basically two competing conditions on $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$. On the one hand $u_{\varepsilon_k}$ is supposed to converge to $u$ with $\nabla u \in \mathcal{N}^{(2)}$ as $k \to \infty$, while on the other hand the limsup inequality (7.38) requires $\nabla u_{\varepsilon_k}$ to be close to $\mathcal{M}^{(2)}$ for large $k \in \mathbb{N}$. In order to meet both of these requirements oscillations have to be incorporated. To this end we make an explicit construction based on local laminates with position-dependent period and orientation taking advantage of the fact that the relaxation of $W = W_2$ was obtained by first-order laminates, see the proof of Theorem 4.1. More technically speaking, we first establish elementary building blocks of the recovery sequence involving simple laminates and then extend the construction globally by covering a fixed volume percentage of the whole domain iteratively with these ball-shaped blocks.
As a start let us prove a sequence of lemmata beginning with the ones that provide the local arguments. The first lemma contains a construction resting upon convex integration as discussed in Section 5.7.

**Lemma 7.21** *Suppose $u_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $u_0(y) = Fy$ with $F \in \mathcal{N}^{(2)}$ and let $\xi \in (0,1)$. Then there are functions $\gamma \in L^\infty(B(0,1))$ and $z \in W^{1,\infty}(B(0,1);\mathbb{R}^2)$ such that*

*(i)* $z = u_0$ *on* $\partial B(0,1)$,

*(ii)* $\nabla z(\mathbb{I} - \gamma s \otimes m) \in \mathrm{SO}(2)$ *almost everywhere in* $B(0,1)$,

*(iii)* $\gamma^2 \leq W^{\mathrm{qc}}(F) + \xi$ *almost everywhere in* $B(0,1)$,

*(iv)* $\|z - u_0\|_{L^1(B(0,1);\mathbb{R}^2)} \leq |B(0,1)|\xi$.

*PROOF.* If $F \in \mathcal{M}^{(2)}$, setting $z = u_0$ and defining $\gamma$ by the unique representation $F = R(\mathbb{I} + \gamma s \otimes m)$ with $R \in \mathrm{SO}(2)$ yields functions $z$ and $\gamma$ satisfying the required properties in view of $\gamma^2 = |Fm|^2 - 1 = W^{\mathrm{qc}}(F)$.
From now on let $F \in \mathcal{N}^{(2)} \setminus \mathcal{M}^{(2)}$. Following the laminate construction in the proof of Theorem 4.1 there are matrices $F^+, F^- \in \mathcal{M}^{(2)}$ whose difference has rank one and $\lambda \in (0,1)$ such that $F = \lambda F^+ + (1-\lambda)F^-$, $F^+ s \neq F^- s$ and $F^+ m = F^- m$. Let $a \in \mathbb{R}^2$ be such that $F^+ - F^- = a \otimes s$. At this point we apply Theorem 5.31 with $v = s$. Hence, for $\delta > 0$ (to be chosen later) there is $h_0 > 0$ such that for $h \in (0, h_0)$ (to be chosen later) there exists a finitely piecewise affine function $v \in W^{1,\infty}(B(0,1); \mathbb{R}^2)$ with $v = u_0$ on $\partial B(0,1)$ satisfying the following properties. There is a set $V \subset B(0,1)$ with $|B(0,1) \setminus V| < \delta$ such that $v$ is equal to a simple laminate between $F^+$ and $F^-$ with weight $\lambda$ and period $h$ on $V$. Moreover, $\nabla v \in \mathcal{N}^{(2)}$ and $\mathrm{dist}(\nabla v, [F^-, F^+]) \leq \delta$ almost everywhere in $B(0,1)$.
For $\xi \in (0,1)$ we define $U^\xi = \{M \in \mathcal{N}^{(2)} : |Mm|^2 < |Fm|^2 + \xi\}$ and assume $\delta$ small enough to satisfy $\delta\,(2|Fm| + \delta) < \xi$. Then the calculation

$$\begin{aligned}
|(\nabla v)m| - |Fm| &= \min_{S \in [F_+, F_-]} \big(|(\nabla v)m| - |Sm|\big) \\
&\leq \min_{S \in [F_+, F_-]} |\nabla v - S| = \mathrm{dist}(\nabla v, [F_+, F_-]) \leq \delta
\end{aligned}$$

implies $|(\nabla v)m|^2 - |Fm|^2 \leq \delta\,(|(\nabla v)m| + |Fm|) \leq \delta\,(2|Fm| + \delta) < \xi$ almost everywhere. All in all this yields $\nabla v \in U^\xi$ almost everywhere in $B(0,1)$.
In the next step we modify the finitely piecewise affine function $v$ by applying Lemma 5.30 to its affine pieces for which $\nabla v \notin \mathcal{M}^{(2)}$ and obtain a function $z \in W^{1,\infty}(B(0,1); \mathbb{R}^2)$ with $z = u_0$ on $\partial B(0,1)$ and $\nabla z \in U^\xi \cap \mathcal{M}^{(2)}$ almost everywhere in $B(0,1)$. Besides, $z$ coincides with $v$ on the set $V$, so that we have for all $y \in V$

$$z(y) = Fy + h\chi_\lambda\left(\frac{s \cdot y}{h}\right) a, \tag{7.47}$$

where $\chi_\lambda$ is a continuous, bounded and one-periodic real-valued function of one variable with mean value zero on $(0,1)$ such that

$$\chi_\lambda'(t) = \begin{cases} 1 - \lambda & \text{if } t \in (0, \lambda), \\ -\lambda & \text{if } t \in (\lambda, 1). \end{cases}$$

Defining $\gamma$ by the unique decomposition $\nabla z = R(\mathbb{I} + \gamma s \otimes m)$ with $R \in \mathrm{SO}(2)$, immediately implies (ii) and (iii) follows from $\gamma^2 = |(\nabla z)m|^2 - 1 \leq |Fm|^2 - 1 + \xi = W^{\mathrm{qc}}(F) + \xi$ almost everywhere. It remains to show (iv). To this end we write

$$\|z - u_0\|_{L^1(B(0,1);\mathbb{R}^2)} = \|z - u_0\|_{L^1(V;\mathbb{R}^2)} + \|z - u_0\|_{L^1(B(0,1)\setminus V;\mathbb{R}^2)}.$$

In view of (7.47) and $|\chi_\lambda(t)| \leq 1$ for all $t \in (0,1)$ one obtains for the first term that

$$\|z - u_0\|_{L^1(V;\mathbb{R}^2)} \leq |a|\,|V|\,h.$$

Regarding the second term we observe that $|\nabla z|^2 = 2+\gamma^2 \leq W^{\mathrm{qc}}(F)+2+\xi \leq W^{\mathrm{qc}}(F)+3$ almost everywhere in $\Omega$. Then the Lipschitz continuity of $z-u_0$ implies that there exists a constant $c>0$ depending only on $F$ such that $|z-u_0| \leq c$ pointwise almost everywhere in $B(0,1)$. Thus, taking $h<\delta$ gives

$$\|z-u_0\|_{L^1(B(0,1);\mathbb{R}^2)} \leq |a|\,|V|h + c\,|B(0,1)\setminus V| \leq \big(|a|\,|B(0,1)| + c\big)\delta.$$

Finally, if $\delta$ is chosen sufficiently small, this concludes the proof. □

The next lemma contains the local construction on balls and hence provides the elementary building blocks, which are the first step towards a global recovery sequence.

**Lemma 7.22** *Let $x \in \mathbb{R}^2$, $\rho > 0$, $u \in W^{1,2}(B(x,\rho);\mathbb{R}^2)$ and $F \in \mathcal{N}^{(2)}$. Assume that*

$$\delta = \frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} |\nabla u - F|^2 \,\mathrm{d}y$$

*satisfies $\delta \in (0,1)$. Then there are functions $\gamma \in L^2(B(x,\rho))$ and $w \in W^{1,2}(B(x,\rho);\mathbb{R}^2)$ such that*

*(i)* $w = u$ *on* $\partial B(x,\rho)$,

*(ii)* $\displaystyle\int_{B(x,\rho)} \gamma^2 \,\mathrm{d}y \leq \int_{B(x,\rho)} W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}y + \sqrt{\delta}\Big(\int_{B(x,\rho)} |\nabla u|^2 \,\mathrm{d}y + 3|B(x,\rho)|\Big)$,

*(iii)* $\displaystyle\int_{B(x,\rho)} \mathrm{dist}^2\big(\nabla w(\mathbb{I} - \gamma s \otimes m), \mathrm{SO}(2)\big)\,\mathrm{d}y \leq |B(x,\rho)|\big(W^{\mathrm{qc}}(F)+3\big)\delta$,

*(iv)* $\|w-u\|_{L^1(B(x,\rho);\mathbb{R}^2)} \leq \rho|B(x,\rho)|\delta$.

*PROOF.* After a scaling and translation argument it is sufficient to consider the problem on $B(0,1)$ and to take $u_{B(0,1)} = 0$. To see this let $u,w : B(x,\rho) \to \mathbb{R}^2$ be given by $u(y) = \rho\tilde{u}(\frac{y-x}{\rho}) + u_{B(x,\rho)}$ and $w(y) = \rho\tilde{w}(\frac{y-x}{\rho}) + u_{B(x,\rho)}$ for all $y \in B(x,\rho)$ and define $\gamma(y) = \tilde{\gamma}(\frac{y-x}{\rho})$ for all $y \in B(x,\rho)$, where $\tilde{u},\tilde{w} \in W^{1,2}(B(0,1);\mathbb{R}^2)$ and $\tilde{\gamma} \in L^2(B(0,1))$ fulfill all the conditions and assertions of Lemma 7.22 for $x=0$ and $\rho=1$. Further, $\tilde{u}_{B(0,1)} = 0$. Since $\delta$ is invariant under this scaling, as well as the term on the left-hand side in (iii) divided by $|B(x,\rho)|$, $w$ satisfies (iii). Explicit calculations show that the scaling yields an additional factor $\rho^2$ in each of the terms of (ii). Finally, this factor cancels so that (ii) holds for $u$ and $\gamma$. Regarding (iv) one finds

$$\begin{aligned}\|w-u\|_{L^1(B(x,\rho);\mathbb{R}^2)} &= \rho\int_{B(x,\rho)} \big|\tilde{w}\big(\tfrac{y-x}{\rho}\big) - \tilde{u}\big(\tfrac{y-x}{\rho}\big)\big|\,\mathrm{d}y \\ &= \rho\,\rho^2\int_{B(0,1)} |\tilde{w}(z)-\tilde{u}(z)|\,\mathrm{d}z \;\leq\; \rho\,\rho^2|B(0,1)|\delta \;=\; \rho\,|B(x,\rho)|\delta.\end{aligned}$$

Here we finish these detailed remarks on scaling and translation. Let us continue the proof working on the unit ball $B(0,1) \subset \mathbb{R}^2$ from now on.
By $\gamma \in L^\infty(B(0,1))$ and $z \in W^{1,\infty}(B(0,1);\mathbb{R}^2)$ we denote the functions resulting from Lemma 7.21 with $\xi = \delta$ and $u_0(y) = Fy$. Let us define

$$w(y) = u(y) + z(y) - u_0(y) \qquad \text{for} \quad y \in B(0,1)\,. \tag{7.48}$$

While (i) and (iv) follow immediately from Lemma 7.21(i) and (iv) by construction, it still remains to show (ii) and (iii). Observe that Lemma 7.21(iii) and Theorem 4.1 give

$$\begin{aligned}
\int_{B(0,1)} \gamma^2 \,\mathrm{d}y &\le \int_{B(0,1)} \big(W^{\mathrm{qc}}(F) + \delta\big)\,\mathrm{d}y \\
&\le \int_{B(0,1)} \big(|(\nabla u)m|^2 - 1 + \delta + |Fm|^2 - |(\nabla u)m|^2\big)\,\mathrm{d}y \\
&\le \int_{B(0,1)} W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}y + \delta|B(0,1)| + 2\sqrt{\delta}\,|B(0,1)| + \sqrt{\delta}\int_{B(0,1)} |\nabla u|^2\,\mathrm{d}y\,.
\end{aligned}$$

For the last inequality we used the algebraic estimate $(a+b)^2 - b^2 = a^2 + 2\,a\cdot b \le (1+\delta^{-1/2})a^2 + \delta^{1/2}b^2$ for all $a,b \in \mathbb{R}^2$ to obtain

$$\begin{aligned}
\int_{B(0,1)} \big(|Fm|^2 - |(\nabla u)m|^2\big)\,\mathrm{d}y &\le \int_{B(0,1)} \big(\tfrac{2}{\sqrt{\delta}}\,|Fm - (\nabla u)m|^2 + \sqrt{\delta}|(\nabla u)m|^2\big)\,\mathrm{d}y \\
&\le 2\sqrt{\delta}\,|B(0,1)| + \sqrt{\delta}\int_{B(0,1)} |\nabla u|^2\,\mathrm{d}y\,.
\end{aligned}$$

Finally, for the verification of (iii), let $R = \nabla z(\mathbb{I}-\gamma s\otimes m) \in \mathrm{SO}(2)$, which is an implication of Lemma 7.21(ii). Then, in view of $|\mathbb{I}-\gamma s\otimes m|^2 = \gamma^2+2 \le W^{\mathrm{qc}}(F)+\delta+2 \le W^{\mathrm{qc}}(F)+3$ one can compute pointwise almost everywhere that

$$\begin{aligned}
\mathrm{dist}^2\big(\nabla w(\mathbb{I}-\gamma s\otimes m), \mathrm{SO}(2)\big) &\le \big|\nabla w(\mathbb{I}-\gamma s\otimes m) - R\big|^2 \\
&\le |\mathbb{I}-\gamma s\otimes m|^2\,|\nabla w - \nabla z|^2 \le (W^{\mathrm{qc}}(F)+3)\,|\nabla u - F|^2.
\end{aligned}$$

Hence,

$$\int_{B(0,1)} \mathrm{dist}^2\big(\nabla w(\mathbb{I}-\gamma s\otimes m), \mathrm{SO}(2)\big)\,\mathrm{d}y \le |B(0,1)|\,(W^{\mathrm{qc}}(F)+3)\delta$$

and the proof of the lemma is complete. □

The following lemma is the crucial tool for the extension of the local construction to a global scale. It raises a technique to use Lemma 7.22 on a fixed volume percentage of those parts of $\Omega$ where no construction as has been applied so far.

**Lemma 7.23** *There exists a $\theta \in (0,1)$ with the following properties: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $G \subset \Omega$ a closed subset. For every $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ with $\nabla u \in \mathcal{N}^{(2)}$ almost everywhere on $\Omega \backslash G$ and every $\xi > 0$ there are a closed set $\omega \subset \Omega \backslash G$ and functions $w \in W^{1,2}(\Omega;\mathbb{R}^2)$ and $\gamma \in L^2(\omega)$ such that*

*(i)* $u = w$ *on* $\Omega \setminus \omega$,

*(ii)* $|\omega| \geq \theta |\Omega \setminus G|$,

*(iii)* $\displaystyle\int_\omega \gamma^2 \,\mathrm{d}y \leq \int_\omega W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}y + \xi|\omega|$,

*(iv)* $\displaystyle\int_\omega \mathrm{dist}^2\big(\nabla w(\mathbb{I} - \gamma s \otimes m), \mathrm{SO}(2)\big)\,\mathrm{d}y \leq \xi|\omega|$,

*(v)* $\|u - w\|_{L^1(\Omega;\mathbb{R}^2)} \leq \xi|\omega|$.

*PROOF.* Suppose $\Omega_L$ denotes the intersection of $\{x \in \Omega \setminus G \,|\, \nabla u(x) \in \mathcal{N}^{(2)}\}$ with the set of $L^2$-Lebesgue points of $\nabla u$ in $\Omega \setminus G$. Further let $\eta \in (0,1)$ to be chosen later. Then we define $\delta : \Omega_L \to (0,1)$ by

$$\delta(x) = \frac{\eta}{W^{\mathrm{qc}}(\nabla u(x)) + 3}\,.$$

In view of the definition of $\Omega_L$ and the fact that $\Omega \setminus G$ is open, one can find for every $x \in \Omega_L$ a $\rho(x) \in (0,1)$ such that $\overline{B(x,\rho(x))} \subset \Omega \setminus G$ and

$$\int_{B(x,\rho(x))} |\nabla u(y) - \nabla u(x)|^2 \,\mathrm{d}y \leq |B(x,\rho(x))|\,\delta(x)\,. \tag{7.49}$$

By Vitali's covering theorem there is an at most countable set $I \subset \Omega_L$ such that the balls of the family $\{\overline{B(x,\rho(x))}\}_{x\in I}$ are pairwise disjoint and satisfy $\sum_{x\in I}|B(x,\rho(x))| \geq \frac{1}{25}|\Omega \setminus G|$. Then, a set of finitely many points $\{x_1,\dots,x_N\} \subset I$ can be chosen in such a way that

$$\sum_{j=1}^N |B(x_j,\rho_j)| \geq \frac{1}{30}|\Omega \setminus G|\,, \tag{7.50}$$

where the notation $\rho_j = \rho(x_j)$ was used. We set $\theta = 1/30$ and

$$\omega = \bigcup_{j=1}^N \overline{B(x_j,\rho_j)}\,,$$

so that (ii) is satisfied. The next step is now to apply Lemma 7.22 to each of the balls $B(x_j,\rho_j)$ with $F=\nabla u(x_j)$ in order to obtain functions $w_j\in W^{1,2}(B(x_j,\rho_j);\mathbb{R}^2)$ and $\gamma_j\in L^2(B(x_j,\rho_j))$ for $j\in\{1,\dots,N\}$. Now define $w\in W^{1,2}(\Omega;\mathbb{R}^2)$ by

$$w(x)=\begin{cases} w_j(x), & \text{if } x\in B(x_j,\rho_j) \text{ for some } j\in\{1,\dots,N\},\\ u(x), & \text{otherwise,}\end{cases}$$

and $\gamma\in L^2(\omega)$ by $\gamma(x)=\gamma_j(x)$, if $x\in B(x_j,\rho_j)$ for some $j\in\{1,\dots,N\}$. By construction (i) holds true and we may compute using (7.49) that

$$\begin{aligned}\int_\omega \gamma^2\,\mathrm{d}y &\leq \sum_{j=1}^N\Big[\int_{B(x_j,\rho_j)}\big(W^{\mathrm{qc}}(\nabla u)+\sqrt{\delta(x_j)}|\nabla u|^2\big)\,\mathrm{d}y+3|B(x_j,\rho_j)|\sqrt{\delta(x_j)}\Big]\\ &\leq \int_\omega W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}y+\sqrt{\eta}\int_\Omega|\nabla u|^2\,\mathrm{d}y+3\sqrt{\eta}|\omega|\,. \end{aligned}\tag{7.51}$$

Furthermore, again with (7.49),

$$\begin{aligned}&\int_\omega \mathrm{dist}^2\big(\nabla w(\mathbb{I}-\gamma s\otimes m),\mathrm{SO}(2)\big)\,\mathrm{d}y\\ &\qquad\leq\sum_{j=1}^N|B(x_j,\rho_j)|\big(W^{\mathrm{qc}}(\nabla u(x_j))+3\big)\delta(x_j)\leq\eta\sum_{j=1}^N|B(x_j,\rho_j)|=|\omega|\eta\end{aligned}$$

and $\|u-w\|_{L^1(\Omega;\mathbb{R}^2)}=\sum_{j=1}^N\|u-w\|_{L^1(B(x_j,\rho_j);\mathbb{R}^2)}\leq\sum_{j=1}^N\rho_j|B(x_j,\rho_j)|\delta(x_j)\leq|\omega|\eta$. The assertions (iii)-(v) of the lemma are immediate, if $\eta$ is chosen sufficiently small. □

In this last step we iterate Lemma 7.23 infinite times until the whole domain $\Omega$ is covered with balls containing the local construction. The key ingredient to guarantee convergence of this procedure is that one can pick in each step the fixed volume percentage $\theta$ of the remaining domain to repeat the construction.

**Lemma 7.24** *Let $\Omega\subset\mathbb{R}^2$ be a bounded and open set. For every $u\in W^{1,2}(\Omega;\mathbb{R}^2)$ with $\nabla u\in\mathcal{N}^{(2)}$ almost everywhere and every $\xi>0$ there are $w\in W^{1,1}(\Omega;\mathbb{R}^2)$ and $\gamma\in L^2(\Omega)$ such that*

*(i)* $\|u-w\|_{L^1(\Omega;\mathbb{R}^2)}\leq\xi|\Omega|$,

*(ii)* $\displaystyle\int_\Omega\gamma^2\,\mathrm{d}y\leq\int_\Omega W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}y+\xi|\Omega|$,

*(iii)* $\displaystyle\int_\Omega\mathrm{dist}^2(\nabla w(\mathbb{I}-\gamma s\otimes m),\mathrm{SO}(2))\,\mathrm{d}y\leq\xi|\Omega|$.

*PROOF.* Using Lemma 7.23 we iteratively construct sequences of functions $w_j$, $\gamma_j$ and of closed sets $\omega_j \subset \Omega$ for $j \in \mathbb{N}$. In the first step we set $G = G_0 = \emptyset$ and take $u$ as in the statement of Lemma 7.24 to get $w_1$, $\omega_1$, $\gamma_1$ satisfying all the properties asserted in Lemma 7.23. In the $(j+1)$-th step with $j \geq 1$ we apply Lemma 7.23 to $w_j$ and $G_j = G_{j-1} \cup \omega_j = \bigcup_{i=1}^{j} \omega_i$.
Observe that Lemma 7.23 (ii) implies

$$|\Omega \setminus G_j| \leq (1-\theta)^j |\Omega| \qquad \text{for } j \in \mathbb{N}. \tag{7.52}$$

Indeed, $|G_j| = |G_{j-1}| + |\omega_j| \geq |G_{j-1}| + \theta|\Omega \setminus G_{j-1}|$ results in $|\Omega \setminus G_j| = |\Omega| - |G_j| \leq |\Omega| - |G_{j-1}| - \theta|\Omega \setminus G_{j-1}| \leq (1-\theta)|\Omega \setminus G_{j-1}|$, which by iteration gives (7.52). As a consequence we have shown $|\Omega \setminus \bigcup_{i=1}^{\infty} \omega_i| = \lim_{j\to\infty} |\Omega \setminus G_j| = 0$, meaning that the union of the disjoint sets $\omega_j$ for $j \in \mathbb{N}$ coincides with $\Omega$ up to a set of measure zero.
So we can define

$$w(x) = \begin{cases} w_j(x), & \text{if } x \in \omega_j \text{ for some } j \in \mathbb{N}, \\ u(x), & \text{otherwise} \end{cases} \tag{7.53}$$

and

$$\gamma(x) = \begin{cases} \gamma_j(x), & \text{if } x \in \omega_j \text{ for some } j \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, by Lemma 7.23 (iii) - (v),

$$\int_\Omega \gamma^2 \, \mathrm{d}y = \sum_{j=1}^{\infty} \int_{\omega_j} \gamma_j^2 \, \mathrm{d}y \leq \sum_{j=1}^{\infty} \Big[ \int_{\omega_j} W^{\mathrm{qc}}(\nabla u) \, \mathrm{d}y + |\omega_j|\xi \Big] = \int_\Omega W^{\mathrm{qc}}(\nabla u) \, \mathrm{d}y + |\Omega|\xi,$$

which implies $\gamma \in L^2(\Omega)$. Further,

$$\begin{aligned} \int_\Omega \mathrm{dist}^2\big(\nabla w(\mathbb{I} - \gamma s \otimes m), \mathrm{SO}(2)\big) \, \mathrm{d}y &= \sum_{j=1}^{\infty} \int_{\omega_j} \mathrm{dist}^2\big(\nabla w_j(\mathbb{I} - \gamma_j s \otimes m), \mathrm{SO}(2)\big) \, \mathrm{d}y \\ &\leq \xi \sum_{j=1}^{\infty} |\omega_j| = |\Omega|\xi \end{aligned} \tag{7.54}$$

and

$$\int_\Omega |u - w| \, \mathrm{d}y = \sum_{j=1}^{\infty} \int_{\omega_j} |u - w| \, \mathrm{d}y \leq \xi \sum_{j=1}^{\infty} |\omega_j| = |\Omega|\xi. \tag{7.55}$$

It remains to verify $w \in W^{1,1}(\Omega;\mathbb{R}^2)$. To this end consider the multiplicative decomposition $\nabla w = [\nabla w(\mathbb{I} - \gamma s \otimes m)](\mathbb{I} + \gamma s \otimes m)$, where we exploited $\mathbb{I} = (\mathbb{I} - \gamma s \otimes m)(\mathbb{I} + \gamma s \otimes m)$. Due to (7.54) the first factor is in $L^2(\Omega;\mathbb{R}^{2\times 2})$, while quadratic integrability of the second factor follows from $\gamma \in L^2(\Omega)$. Hence, $\nabla w \in L^1(\Omega;\mathbb{R}^{2\times 2})$ by Hölder's inequality.

Accounting for (7.55) finally provides $w \in W^{1,1}(\Omega;\mathbb{R}^2)$. □

*PROOF of Theorem 7.18 (Upper bound inequality).* The upper bound in Theorem 7.18 is an immediate implication of Lemma 7.24. To see this let $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ with $u_\Omega = 0$ and $\nabla u \in \mathcal{N}^{(2)}$ almost everywhere be given. For $\varepsilon > 0$ we set $\xi = \xi(\varepsilon) = \varepsilon^2$. Then the functions $w_\varepsilon \in W^{1,1}(\Omega;\mathbb{R}^2)$ and $\gamma_\varepsilon \in L^2(\Omega)$ resulting from Lemma 7.24 satisfy

$$\begin{aligned}\int_\Omega W_\varepsilon(\nabla w_\varepsilon)\,\mathrm{d}y &= \frac{1}{\varepsilon}\int_\Omega \mathrm{dist}^2\left(\nabla w_\varepsilon(\mathbb{I}-\gamma_\varepsilon s\otimes m),\mathrm{SO}(2)\right)\mathrm{d}y + \int_\Omega \gamma_\varepsilon^2\,\mathrm{d}y \\ &\leq \frac{1}{\varepsilon}\,\varepsilon^2|\Omega| + \int_\Omega W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}y + \varepsilon^2\,|\Omega| \\ &= \int_\Omega W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}y + |\Omega|\varepsilon(1+\varepsilon).\end{aligned} \tag{7.56}$$

For any subsequence $\{w_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{w_\varepsilon\}_{\varepsilon>0}$ with $\varepsilon_k \to 0$ as $k\to\infty$ we infer from Lemma 7.24 (i) that $w_{\varepsilon_k} \to u$ in $L^1(\Omega;\mathbb{R}^2)$. By restricting (7.56) to the subsequence $\{w_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and by passing to the limit $k\to\infty$ one finds $\limsup_{k\to\infty}\int_\Omega W_{\varepsilon_k}(\nabla w_{\varepsilon_k})\,\mathrm{d}y \leq \int_\Omega W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}y$. In order to finish the proof we define $u_{\varepsilon_k} = w_{\varepsilon_k} - (w_{\varepsilon_k})_\Omega$ for $k\in\mathbb{N}$, so that $(u_{\varepsilon_k})_\Omega = 0$. Eventually, $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is the desired recovery sequence. □

**Remark 7.25** *1. Let us remark that there is another proof of the upper bound inequality doing without convex integration. Indeed, Lemma 7.21 can be replaced by an explicit and elementary construction where the laminate resulting from the relaxation of $W$ is simply cut off close to the boundary of the unit ball to achieve the required affine boundary conditions. However, this alternative procedure can only guarantee $\nabla z \in \mathcal{M}^{(2)}$ on balls $B(0,r)$ with $r < 1$. In order to make up for this drawback careful estimates in the remaining annulus are necessary.*
*We will apply this direct approach later in Section 8 to establish a recovery sequence for the $\Gamma$-convergence result of Theorem 8.3 in the three-dimensional setting. Notice that in 3D the situation is more rigid and Lemma 5.31 does not hold, so that we are reliant on arguments avoiding convex integration there.*
*2. In Theorem* 7.18 *all admissible deformations were supposed to have mean value zero. Instead of postulating such an average condition one can as well prescribe boundary data to make the problem well-posed. It is a matter of the model at hand and the purpose in mind which alternative is favorable. Focusing on a single time step in a biaxial shear test, for instance, necessitates the type with fixed boundary data, while the approach via mean values should be used, if one is interested in learning how a specimen changes its outer shape in response to external forces.*
*With respect to an exact mathematical reformulation of Theorem 7.18 we need to replace the assumption that $(u_\varepsilon)_\Omega = 0$ for a bounded energy sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of $E_\varepsilon$ by the requirement $u_\varepsilon \in W^{1,1}_v(\Omega;\mathbb{R}^2)$ for all $\varepsilon > 0$ with given $v \in W^{1,2}(\Omega;\mathbb{R}^2)$. Notice the technical motivation for either of these two properties relates to the application of Poincaré's*

*inequality in order to extract a convergent subsequence of* $\{u_\varepsilon\}_{\varepsilon>0}$.
*Regarding the proof of Theorem 7.18 with boundary conditions there are two minor changes we want to comment on briefly. The other arguments remain perfectly the same. In the proof of compactness, we exploit that* $W^{1,1}_v(\Omega;\mathbb{R}^2)$ *is a weakly closed subset of* $W^{1,1}(\Omega;\mathbb{R}^2)$*, so that the boundary conditions of the sequence* $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ *carry over to its weak* $W^{1,1}$*-limit* $u$. *Concerning the upper bound inequality and an appropriate recovery sequence the way* $w$ *is defined by* (7.53) *in the proof of Lemma 7.24, where we assume* $u \in W^{1,2}_v(\Omega;\mathbb{R}^2)$*, shows that the boundary conditions are preserved by the construction.*

## 7.4. Generalizations

The results that have been achieved so far in this chapter are now extended to general growth exponents and adapted for more realistic elastic energy densities.

### 7.4.1. General elastic and plastic growth exponents

This section is devoted to the analysis of the energy densities $W_\varepsilon = W_{\varepsilon;q,p}$ from (7.1) for general elastic and plastic growth exponents $q \geq 1$ and $p \geq 1$. In particular, we want to investigate asymptotic behavior of these models with increasing penalization of elastic energy. As the findings of Section 7.2 and 7.3 show, limiting behavior depends qualitatively on whether we face a system with or without hardening. Here we want to study the effect of the intensity of elasticity and hardening by deciding which choice of growth exponents leads to what type of material response. All the results presented here can be understood as generalizations of what has been proven up to now.

#### The threshold of soft material behavior

The next result extends Theorem 7.10, which gives a statement for the special choice $p = 1$ and $q = 2$.

**Theorem 7.26** *Let* $\varepsilon > 0$. *If* $p > 1$ *and* $1 \leq q < \frac{p}{p-1}$ *or* $p = 1$ *and* $q \geq 1$*, then it holds* $W^{\mathrm{pc}}_\varepsilon(F) = W^{\mathrm{qc}}_\varepsilon(F) = W^{\mathrm{rc}}_\varepsilon(F) = 0$ *for all* $F \in \mathcal{N}^{(2)}$.

*PROOF.* The argumentation proceeds along the lines of the proof of Theorem 7.10. Again it is sufficient to restrict attention to matrices of the form $F_* = \mathbb{I} + \sigma\gamma_0 s \otimes m \in \mathcal{M}^{(2)}$ with $\gamma_0 > 0$ and $\sigma \in \{-1, 1\}$. Instead of defining $R_r$ by (7.14) we take here $R_r = (r-1)s \otimes s + \sigma r^\alpha s \otimes m$ with $r > \max\{1, \gamma_0^{1/\alpha}\}$ and $\alpha \in \mathbb{R} \setminus \{0\}$ (to be chosen later). In analogy to (7.15) one obtains a family of rank-one lines $t \to F_r(t)$ such that $F_* = F_r(\gamma_0 r^{-\alpha}) = (1-\lambda_r)F_r(0) + \lambda_r F_r(1)$ with $\lambda_r = \gamma_0 r^{-\alpha}$. Following the ideas of (7.17) and (7.18) we can estimate $W_\varepsilon(F_r(0)) \leq c(\gamma_0, \varepsilon) r^{-\alpha-1}$ and $W_\varepsilon(F_r(1)) \leq c(\gamma_0, \varepsilon)\big(1 + r^{\max\{q,(\alpha-1)p\}}\big)$. In order to show

$$W^{\mathrm{rc}}_\varepsilon(F_*) \leq W_\varepsilon\big(F_r(0)\big) + \lambda_r W_\varepsilon\big(F_r(1)\big) \longrightarrow 0$$

as $r \to \infty$, the parameter $\alpha \in \mathbb{R} \setminus \{0\}$ has to meet the requirement $\alpha > \max\{q, (\alpha - 1)p\}$. Such an $\alpha$ exists if and only if $p$ and $q$ satisfy the properties stated in the theorem. □

### Pointwise convergence of envelopes

The result of Theorem 7.12 can be generalized to the setting with $q \geq 2$ as follows.

**Theorem 7.27** *Let $p = 2$, $q \geq 2$ and $F \in \mathbb{R}^{2\times 2}$. Then,*

$$\lim_{\varepsilon\to 0} W_\varepsilon^{\mathrm{qc}}(F) = W^{\mathrm{qc}}(F).$$

*The same statement holds true, if $W_\varepsilon^{\mathrm{qc}}$ is replaced by the polyconvex or rank-one convex envelopes $W_\varepsilon^{\mathrm{pc}}$ and $W_\varepsilon^{\mathrm{rc}}$.*

*PROOF.* The case $p = q = 2$ has already been proven in Theorem 7.12. So we may assume $q > 2$. Using Young's inequality it can be shown that $a^q \geq a^2\varepsilon^{\frac{q-1}{q}} - (q/2 - 1)(2/q)^{\frac{q}{q-2}}\varepsilon^{\frac{q-1}{q-2}}$ for all $a \in \mathbb{R}_0^+$ and $\varepsilon > 0$. Hence,

$$\begin{aligned} W_{\varepsilon;q,2}(F) &\geq \inf_{\gamma\in\mathbb{R}}\left\{\frac{1}{\varepsilon^{1/q}}\operatorname{dist}^2(F(\mathbb{I} - \gamma s\otimes m), \mathrm{SO}(2)) + |\gamma|^2\right\} - (q/2-1)(2/q)^{\frac{q}{q-2}}\varepsilon^{\frac{1}{q-2}} \\ &= W_{\varepsilon^{1/q};2,2}(F) - (q/2-1)(2/q)^{\frac{q}{q-2}}\varepsilon^{\frac{1}{q-2}}. \end{aligned}$$

In view of Theorem 7.12 we deduce

$$\begin{aligned} \lim_{\varepsilon\to 0} W_{\varepsilon;q,2}^{\mathrm{qc}}(F) &\geq \lim_{\varepsilon\to 0}\left(W_{\varepsilon^{1/q};2,2}^{\mathrm{qc}}(F) - (q/2-1)(2/q)^{\frac{q}{q-2}}\varepsilon^{\frac{1}{q-2}}\right) \\ &= \lim_{\varepsilon\to 0} W_{\varepsilon;2,2}^{\mathrm{qc}}(F) = W_2^{\mathrm{qc}}(F). \end{aligned}$$

The reverse inequality follows, since $W_{\varepsilon;q,2}^{\mathrm{qc}} \leq W_2^{\mathrm{qc}}$ by Lemma 7.2 (ii). This yields the assertion regarding convergence of the quasiconvex envelopes. The same reasoning holds for the rank-one and polyconvex envelopes. □

### A generalized approximation result

Provided the plastic and elastic exponents of the condensed energy density $W_\varepsilon = W_{\varepsilon;q,p}$ satisfy $p \geq 2$ and $2 \leq q \leq 2p$ we are able to prove a $\Gamma$-convergence result in the style of Theorem 7.18. More precisely, we will show that the limit of the energy functionals $E_\varepsilon = E_{\varepsilon;q,p}$ is characterized by the relaxation of the elastically rigid energy density $W = W_p$. Notice that the gradients of finite energy sequences of $E_\varepsilon$ are in their substantial part bounded in $L^p$, while the corresponding determinants converge strongly in $L^{q/2}(\Omega)$. These observations in combination with the availability of relaxations for $W$ and the result of Theorem 7.26 motivate the lower bounds on $p$ and $q$. The necessity of restricting $q$ from above is technical in nature and arises in the construction of the recovery sequence when proving the upper bound.

**Theorem 7.28** *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded and open set with Lipschitz boundary and $p \geq 2$, $2 \leq q \leq 2p$. For $\varepsilon > 0$ let $E_\varepsilon$ and $E$ be the functionals defined in (7.7) and (7.8), respectively. Then, $E_\varepsilon$ converge in the sense of $\Gamma$–convergence to the functional $E$ with respect to strong convergence in $L^{\frac{pq}{p+q}}$ as $\varepsilon$ tends to zero. Moreover, any bounded energy sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of $E_\varepsilon$ is relatively compact in $L^{\frac{pq}{p+q}}$. In detail,*

Compactness and lower bound inequality: *Suppose that $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,1}(\Omega;\mathbb{R}^2)$ satisfies $(u_\varepsilon)_\Omega = 0$ for all $\varepsilon > 0$ and is a sequence of bounded energy in the sense that there is a constant $B < \infty$ such that $E_\varepsilon[u_\varepsilon] < B$ for all $\varepsilon > 0$. Then there exist a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ as $k \to \infty$ and a function $u \in W^{1,p}(\Omega;\mathbb{R}^2)$ such that $u_{\varepsilon_k} \to u$ in $L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^2)$ with $u_\Omega = 0$, $|(\nabla u)s| \leq 1$ and $\det \nabla u = 1$ almost everywhere in $\Omega$. Moreover, it holds*

$$E[u] \leq \liminf_{k\to\infty} E_{\varepsilon_k}[u_{\varepsilon_k}]\,.$$

Upper bound inequality: *For every $u \in W^{1,p}(\Omega;\mathbb{R}^2)$ with $u_\Omega = 0$ and $\nabla u \in \mathcal{N}^{(2)}$ almost everywhere in $\Omega$ there exists a sequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}} \subset W^{1,\frac{pq}{p+q}}(\Omega;\mathbb{R}^2)$ with $(u_{\varepsilon_k})_\Omega = 0$ for all $k \in \mathbb{N}$ and $u_{\varepsilon_k} \to u$ in $L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^2)$ as $k \to \infty$ such that*

$$E[u] \geq \limsup_{k\to\infty} E_{\varepsilon_k}[u_{\varepsilon_k}]\,. \tag{7.57}$$

The basic ideas of the proof are to a large extent analogous to the ones used to show Theorem 7.18. Subsequently, we subdivide the proof again into its natural three steps and point out the main differences with respect to the situation of linear hardening and quadratic elastic energy discussed in Section 7.3.2. Actually, for the construction of the recovery sequence in the case $q > p$ a refined argumentation is necessary, so that we will need to go into the details there.

### Compactness

In view of the definitions of Section 7.1.3 the generalized version of Lemma 7.19 reads as follows.

**Lemma 7.29** *Suppose $p \geq 2$, $2 \leq q \leq 2p$. Let $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,1}(\Omega;\mathbb{R}^2)$ be a sequence with $E_\varepsilon[u_\varepsilon] < B$ for all $\varepsilon > 0$. Then,*

*(i)* $\|a_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^2)} \leq 1$,

*(ii)* $\|g_\varepsilon^{(1)}\|_{L^q(\Omega;\mathbb{R}^2)} \leq (\varepsilon B)^{1/q}$

*(iii)* $\|g_\varepsilon^{(2)}\|_{L^p(\Omega;\mathbb{R}^2)} \leq 2^{\frac{p-2}{2p}}(B + |\Omega|)^{1/p}$

*(iv)* $\|h_\varepsilon\|_{L^q(\Omega;\mathbb{R}^2)} \leq (\varepsilon B)^{1/q}$,

*(v)* $\|f_\varepsilon\|_{L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^2)} \leq \varepsilon^{1/q} B^{\frac{p+q}{pq}}$.

*Then, $\{A_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^{\min\{p,q\}}(\Omega;\mathbb{R}^{2\times 2})$ and $\{H_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^{2\times 2})$ and equi-integrable.*

*PROOF.* Most of the proof is in full analogy with the arguments applied to verify Lemma 7.19. Hence, (i), (ii) and (iv) are a direct consequence of (7.4), while (v) results from Hölder's inequality in combination with (iv) and the estimate $\|\gamma_\varepsilon\|_{L^p(\Omega)} \leq B^{1/p}$. Further we compute

$$|g_\varepsilon^{(2)}|^p = |a_\varepsilon^\perp + \gamma_\varepsilon a_\varepsilon|^p = (1+\gamma_\varepsilon^2)^{p/2} \leq 2^{\frac{p-2}{2}}\big(1 + W_\varepsilon(\nabla u_\varepsilon)\big).$$

This proves (iii). $\square$

**Corollary 7.30** *Under the assumptions of Lemma 7.29 there exist a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and functions $a \in L^\infty(\Omega;\mathbb{R}^2)$ and $g \in L^p(\Omega;\mathbb{R}^2)$ such that*

*(i)* $a_{\varepsilon_k} \overset{*}{\rightharpoonup} a$ *in* $L^\infty(\Omega;\mathbb{R}^2)$,

*(ii)* $g_{\varepsilon_k}^{(1)} \to 0$ *in* $L^q(\Omega;\mathbb{R}^2)$,

*(iii)* $g_{\varepsilon_k}^{(2)} \rightharpoonup g$ *in* $L^p(\Omega;\mathbb{R}^2)$,

*(iii)* $h_{\varepsilon_k} \to 0$ *in* $L^q(\Omega;\mathbb{R}^2)$,

*(iv)* $f_{\varepsilon_k} \to 0$ *in* $L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^2)$

*as $k \to \infty$.*
*It holds $H^k := H_{\varepsilon_k} \to 0$ in $L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^{2\times 2})$ and $A^k := A_{\varepsilon_k} \rightharpoonup A$ in $L^{\min\{p,q\}}(\Omega;\mathbb{R}^{2\times 2})$, where $A = a\otimes s + g \otimes m \in L^p(\Omega;\mathbb{R}^{2\times 2})$.*

*PROOF of Theorem 7.28 (Compactness).* Based on the adapted energy estimates of Lemma 7.29, the required compactness for Theorem 7.28 can be shown along the same line of reasoning as in the special case $p = q = 2$ discussed in Section 7.3.2. In fact, if $pq/(p+q) > 1$, the arguments that lead to a converging subsequence are even easier, since one has weak compactness in $W^{1,\frac{pq}{p+q}}$ right away without having to argue with the equi-integrability of $\{\nabla u_\varepsilon\}_{\varepsilon>0}$. Notice that for the proof that $\det \nabla u = 1$ almost everywhere in $\Omega$ one can even take literally the same proof, because $p, q \geq 2$. $\square$

### The lower bound

*PROOF of Theorem 7.28 (Lower bound inequality).* Let $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and $u$ be as in the statement of the theorem and take $\delta > 0$. Then one can find a subsequence $\{u_{\varepsilon_h}\}_{h\in\mathbb{N}}$ such that

$$E_{\varepsilon_h}[u_{\varepsilon_h}] \leq B_\delta := \liminf_{k\to\infty} E_{\varepsilon_k}[u_{\varepsilon_k}] + \delta.$$

As a consequence of Corollary 7.30 it holds, possibly after extracting a further subsequence, again denoted $\{u_{\varepsilon_h}\}_{h\in\mathbb{N}}$, that $g_{\varepsilon_h}^{(2)} \rightharpoonup g = (\nabla u)m$ in $L^p(\Omega;\mathbb{R}^2)$ as $h \to \infty$.

Since the mapping $\varphi : \mathbb{R}^2 \to [0,\infty)$, $x \mapsto \max\{|x|^2-1,0\}^{p/2}$ is convex, the functional $I[v] = \int_\Omega \max\{|v|^2-1,0\}^{p/2}\,\mathrm{d}x$ is weakly lower semicontinuous with respect to $L^p(\Omega;\mathbb{R}^2)$. Hence, in view of the fact that $|(\nabla u)m| \geq 1$ almost everywhere in $\Omega$, which results from $\nabla u \in \mathcal{N}^{(2)}$ almost everywhere, we conclude

$$\begin{aligned} E[u] &= \int_\Omega \big(|(\nabla u)m|^2-1\big)^{p/2}\,\mathrm{d}x = \int_\Omega \varphi\big((\nabla u)m\big)\,\mathrm{d}x \\ &\leq \liminf_{h\to\infty} \int_\Omega \varphi(g^{(2)}_{\varepsilon_h})\,\mathrm{d}x = \liminf_{h\to\infty}\int_\Omega |\gamma_{\varepsilon_h}|^p\,\mathrm{d}x \leq \liminf_{h\to\infty}\int_\Omega W_{\varepsilon_h}(\nabla u_{\varepsilon_h})\,\mathrm{d}x \leq B_\delta. \end{aligned}$$

The parameter $\delta > 0$ being arbitrary proves the theorem. $\square$

### The upper bound

As far as the proof of the upper bound is concerned the construction of the recovery sequence for the growth exponents $p \geq 2$ and $2 \leq q \leq 2p$ requires a more subtle treatment. The basic laminates are constructed similarly to the case $p = q = 2$, see Lemma 7.21. For gaining the elementary building blocks, however, the oscillations $z$ cannot simply be stuck on $u$ as it is done in (7.48), if $q > p$. The reason is that the elastic energy, which has growth of order $q$, cannot be controlled in terms of $\nabla u \in L^p$ any longer. Instead, we pass to a slightly smaller ball for the local construction and make use of harmonic extension methods and elliptic $L^p$-theory to achieve higher integrability. This is the point where the technical result of Lemma 6.3 comes in essentially. When passing to the global construction we use in each iteration step only a fixed part of the original balls, which leads to a change in the value of the volume percentage $\theta$ but does not affect the other arguments. The case $q \leq p$ is much easier and follows exactly the lines of Theorem 7.18. Therefore we concentrate mainly on $q > p$ in the following.
Let us start by an immediate extension of the simple laminate construction in Lemma 7.21 adapted to the setting with general growth exponents $p$ and $q$. After scaling the proof stays the same up to minor changes in showing (iv).

**Lemma 7.31** *Let $r > 0$ and $p, q \geq 2$. Suppose $u_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $u_0(y) = Fy$ with $F \in \mathcal{N}^{(2)}$ and let $\xi \in (0,1)$. Then there are functions $\gamma \in L^\infty(B(0,r))$ and $z \in W^{1,\infty}(B(0,r);\mathbb{R}^2)$ such that*

*(i)* $z = u_0$ *on* $\partial B(0,r)$,

*(ii)* $\nabla z(\mathbb{I} - \gamma s \otimes m) \in \mathrm{SO}(2)$ *almost everywhere in* $B(0,r)$,

*(iii)* $|\gamma|^p \leq \big(W_2^{\mathrm{qc}}(F) + \xi\big)^{p/2}$ *almost everywhere in* $B(0,r)$,

*(iv)* $\|z - u_0\|_{L^{\frac{pq}{p+q}}(B(0,r);\mathbb{R}^2)} \leq r\,|B(0,r)|^{\frac{p+q}{pq}}\,\xi$.

The major difference to the proof of Theorem 7.18 lies in the verification of the corresponding version of Lemma 7.22, which is given right below.

**Lemma 7.32** *Let $p \geq 2$, $2 \leq q \leq 2p$, $x \in \mathbb{R}^2$, $\rho > 0$, $u \in W^{1,p}(B(x,\rho);\mathbb{R}^2)$, $F \in \mathcal{N}^{(2)}$ and $\alpha \in (0,1)$. Assume that*

$$\delta = \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} |\nabla u - F|^p \, \mathrm{d}y \tag{7.58}$$

*satisfies $\delta \in (0,1)$.*
*Then there are $r \in (1/2\rho, 3/4\rho)$, $\gamma \in L^p(B(x,r))$ and $w \in W^{1,p}(B(x,r);\mathbb{R}^2)$ such that*

*(i)* $w = u$ *on* $\partial B(x,r)$,

*(ii)* $$\int_{B(x,r)} |\gamma|^p \, \mathrm{d}y \leq \alpha^{(1-\frac{p}{2})} \int_{B(x,r)} W_p^{\mathrm{qc}}(\nabla u) \, \mathrm{d}y + c(p)\, (1-\alpha)^{(1-\frac{p}{2})} \Big( \int_{B(x,r)} |\nabla u|^p \, \mathrm{d}y + |B(x,r)| \Big) \sqrt{\delta},$$

*(iii)* $$\int_{B(x,r)} \mathrm{dist}^q\big(\nabla w(\mathbb{I} - \gamma s \otimes m), \mathrm{SO}(2)\big) \, \mathrm{d}y \leq c(p,q)\, |B(x,r)| \, \big(W_p^{\mathrm{qc}}(F)^{q/p} + 1\big)\, \delta^{q/p},$$

*(iv)* $$\|w - u\|_{L^{\frac{pq}{p+q}}(B(x,r);\mathbb{R}^2)} \leq c(p,q)\, r\, |B(x,r)|^{\frac{p+q}{pq}} \, \delta^{1/p}.$$

*PROOF.* By scaling and shifting in the independent variables and by translation with respect to the range of $u$ we may assume $B(x,\rho) = B(0,1)$ and $u_{B(0,1)} = 0$ without loss of generality. Moreover, from now on we take $q > p$. The case $q \leq p$ is a lot easier, because all the effort put into gaining higher integrability is not necessary there and one can proceed as in the proof of Theorem 7.18.
Let $u_0(y) = Fy$ for $y \in B(0,1)$. According to Lemma 6.1 we can choose $r \in (1/2, 3/4)$ in such a way that $u - u_0 \in W^{1,p}(\partial B(0,r);\mathbb{R}^2)$ together with the estimate

$$\|u - u_0\|^p_{W^{1,p}(\partial B(0,r);\mathbb{R}^2)} \leq c(p)\, \|\nabla u - F\|^p_{L^p(B(0,1);\mathbb{R}^{2\times 2})} \leq c(p)\, \delta. \tag{7.59}$$

By $\gamma \in L^\infty(B(0,r))$ and $z \in W^{1,\infty}(B(0,r);\mathbb{R}^2)$ we denote the functions constructed in Lemma 7.31 on $B(0,r)$ with $\xi = \delta^{1/p}$. Further, $v$ stands for the harmonic extension of $u$ restricted to $\partial B(0,r)$ to $B(0,r)$. We define $w \in W^{1,p}(B(0,r);\mathbb{R}^2)$ by

$$w(y) = v(y) - u_0(y) + z(y)$$

for $y \in B(0,r)$. This implies (i) in view of Lemma 7.31 (i). Inequality (ii) follows from calculations similar to those in the proof of Lemma 7.22 and the general estimate

$(a+b)^s \leq \alpha^{1-s}a^s + (1-\alpha)^{1-s}b^s$ for all $a, b \in \mathbb{R}_0^+$, $s \geq 1$ and $\alpha \in (0,1)$. More precisely, Lemma 7.31 (iii) gives

$$\begin{aligned}
\int_{B(0,r)} |\gamma|^p \, \mathrm{d}y &= \int_{B(0,r)} \big(W_2^{\mathrm{qc}}(F) + \delta^{1/p}\big)^{p/2} \, \mathrm{d}y \\
&= \int_{B(0,r)} \Big(|(\nabla u)m|^2 - 1 + \delta^{1/p} + |Fm|^2 - |(\nabla u)m|^2\Big)^{p/2} \, \mathrm{d}y \\
&\leq \alpha^{(1-\frac{p}{2})} \int_{B(0,r)} W_p^{\mathrm{qc}}(\nabla u) \, \mathrm{d}y \\
&\quad + (1-\alpha)^{(1-\frac{p}{2})} 2^{(\frac{p}{2}-1)} \cdot \left( \delta^{1/2} |B(0,r)| + \int_{B(0,r)} \big||Fm|^2 - |(\nabla u)m|^2\big|^{p/2} \, \mathrm{d}y \right).
\end{aligned}$$

In order to deal with the last expression, we use $(a+b)^2 - b^2 = a^2 + 2a \cdot b \leq (1 + \frac{1}{\delta^{1/p}})a^2 + \delta^{1/p}b^2 \leq \frac{2}{\delta^{1/p}}a^2 + \delta^{1/p}b^2$ for all $a, b \in \mathbb{R}^2$ and obtain

$$\big||Fm|^2 - |(\nabla u)m|^2\big|^{p/2} \quad \leq \quad \frac{2^{p-1}}{\delta^{1/2}} |F - \nabla u|^p + 2^{(\frac{p}{2}-1)} \delta^{1/2} |\nabla u|^p.$$

So one finally infers (ii) from

$$\int_{B(0,r)} \big||Fm|^2 - |(\nabla u)m|^2\big|^{p/2} \, \mathrm{d}y \leq \delta^{1/2} \Big( 2^{p-1} |B(0,r)| + 2^{(\frac{p}{2}-1)} \int_{B(0,r)} |\nabla u|^p \, \mathrm{d}y \Big).$$

Before starting to verify (iii) we deduce from Lemma 6.3 with $n = 2$ that

$$\|v - u_0\|_{W^{1,2p}(B(0,r);\mathbb{R}^2)} \leq c \, \|u - u_0\|_{W^{1,p}(\partial B(0,r);\mathbb{R}^2)}, \tag{7.60}$$

where $c$ depends on $p$ and $r$. Due to $r \in (1/2, 3/4)$ we actually have $c = c(p)$. This result is crucial for estimating the left-hand side of (iii). Following the ideas of Lemma 7.22 we find

$$\mathrm{dist}\big(\nabla w(\mathbb{I} - \gamma s \otimes m), \mathrm{SO}(2)\big) \leq \big|\nabla w(\mathbb{I} - \gamma s \otimes m) - Q\big| \leq |\mathbb{I} - \gamma s \otimes m| \, |\nabla v - F|$$

in $B(0,r)$, where $Q = \nabla z(\mathbb{I} - \gamma s \otimes m) \in SO(2)$ by Lemma 7.31 (ii). Further,

$$|\mathbb{I} - \gamma s \otimes m|^q \leq \big(W_p^{\mathrm{qc}}(F)^{2/p} + 3\big)^{q/2} \leq c(q) \big(W_p^{\mathrm{qc}}(F)^{q/p} + 1\big).$$

Then, with (7.60) and (7.59) one concludes in view of $q \leq 2p$ and $r \in (1/2, 3/4)$,

$$\begin{aligned}
\|\nabla v - F\|^q_{L^q(B(0,r);\mathbb{R}^{2\times 2})} &\leq c(p,q) \, \|v - u_0\|^q_{W^{1,2p}(B(0,r);\mathbb{R}^2)} \\
&\leq c(p,q) \, \|u - u_0\|^q_{W^{1,p}(\partial B(0,r);\mathbb{R}^2)} \leq c(p,q) \, |B(0,r)| \, \delta^{q/p},
\end{aligned}$$

which is the final ingredient to show (iii). It remains to verify (iv). Here we exploit Lemma 7.31 (iv), Poincaré's and Hölder's inequality and (7.60) to get

$$\begin{aligned} \|w-u\|_{L^{\frac{pq}{p+q}}(B(0,r);\mathbb{R}^2)} &\leq \|z-u_0\|_{L^{\frac{pq}{p+q}}(B(0,r);\mathbb{R}^2)} \\ &\quad + \|u-u_0\|_{L^{\frac{pq}{p+q}}(B(0,r);\mathbb{R}^2)} + \|v-u_0\|_{L^{\frac{pq}{p+q}}(B(0,r);\mathbb{R}^2)} \\ &\leq |B(0,r)|^{\frac{p+q}{pq}}\ \delta^{1/p} + c(p,q)\ \|\nabla u - F\|_{L^p(B(0,1);\mathbb{R}^{2\times 2})} + c(p,q)\ \|v-u_0\|_{W^{1,2p}(B(0,r);\mathbb{R}^2)} \\ &\leq c(p,q)\ r\ |B(0,r)|^{\frac{p+q}{pq}}\ \delta^{1/p}. \end{aligned} \tag{7.61}$$

This concludes the proof. $\square$

Next we state the generalizations of Lemma 7.23 and Lemma 7.24, the proofs of which are very close to the ones in Section 7.3.2. For this reason we only point out the main differences.

**Lemma 7.33** *Assume $p \geq 2$ and $2 \leq q \leq 2p$. There exists $\theta \in (0,1)$ with the following properties: Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set and $G \subset \Omega$ a closed subset. For every $u \in W^{1,p}(\Omega;\mathbb{R}^2)$ with $\nabla u \in \mathcal{N}^{(2)}$ almost everywhere on $\Omega \setminus G$, every $\beta > 1$ and every $\xi > 0$ there are a closed set $\omega \subset \Omega \setminus G$, a function $w \in W^{1,p}(\Omega;\mathbb{R}^2)$ and $\gamma \in L^p(\omega)$ such that*

*(i)* $u = w$ *on* $\Omega \setminus \omega$,

*(ii)* $|\omega| \geq \theta|\Omega \setminus G|$,

*(iii)* $\displaystyle\int_\omega |\gamma|^p \,\mathrm{d}y \leq \beta \int_\omega W_p^{\mathrm{qc}}(\nabla u)\,\mathrm{d}y + \xi\ |\omega|$,

*(iv)* $\displaystyle\int_\omega \operatorname{dist}^q\big(\nabla w(\mathbb{I} - \gamma s \otimes m), \mathrm{SO}(2)\big)\,\mathrm{d}y \leq \xi\ |\omega|$,

*(v)* $\|u-w\|^{\frac{pq}{p+q}}_{L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^2)} \leq \xi\ |\omega|$.

*PROOF.* The main difference with respect to Lemma 7.23 is the choice of the closed subset $\omega \subset \Omega\setminus G$, where the local construction is performed. Here we take $\omega = \bigcup_{j=1}^N B(x_j, r_j)$ with $r_j$ the radii resulting from Lemma 7.32 applied to the balls $B(x_j,\rho_j)$ for $j \in \{1,\ldots,N\}$. From $r_j \geq \rho_j/2$ and (7.50) it follows that $\theta = 1/120$. So in comparison with Lemma 7.23 we have a different value of the fixed volume percentage. This however does not affect the other steps of the proof. Another minor difference to Lemma 7.23 is that one has to account for $\alpha \in (0,1)$, which occurs in Lemma 7.32 (ii), and relate it to

$\beta > 1$. The generalization of (7.51) reads

$$\begin{aligned}
\int_\omega |\gamma|^p \, \mathrm{d}y \;\; &\leq \;\; \sum_{j=1}^N \Bigg[ \alpha^{(1-\frac{p}{2})} \int_{B(x_j,r_j)} W_p^{\mathrm{qc}}(\nabla u)\,\mathrm{d}y \\
&\qquad + c(p)\,(1-\alpha)^{(1-\frac{p}{2})} \sqrt{\delta(x_j)} \Big( \int_{B(x_j,r_j)} |\nabla u|^p \,\mathrm{d}y + |B(x_j,r_j)| \Big) \Bigg] \\
&\leq \;\; \alpha^{(1-\frac{p}{2})} \int_\omega W_p^{\mathrm{qc}}(\nabla u)\,\mathrm{d}y + \; c(p)\,(1-\alpha)^{(1-\frac{p}{2})} \eta^{1/2} \left( \int_\Omega |\nabla u|^p \,\mathrm{d}y + |\omega| \right).
\end{aligned}$$

Choosing $\alpha$ close enough to 1, so that $\alpha^{(1-\frac{p}{2})} \leq \beta$ and taking $\eta$ sufficiently small yields (iii). $\square$

**Lemma 7.34** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $p \geq 2$, $2 \leq q \leq 2p$. For every $u \in W^{1,p}(\Omega;\mathbb{R}^2)$ such that $\nabla u \in \mathcal{N}^{(2)}$ almost everywhere in $\Omega$, every $\beta > 1$ and every $\xi > 0$ there are functions $w \in W^{1,\frac{pq}{p+q}}(\Omega;\mathbb{R}^2)$ and $\gamma \in L^p(\Omega)$ such that*

*(i)* $\|u - w\|^{\frac{pq}{p+q}}_{L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^2)} \leq \xi\,|\Omega|$,

*(ii)* $\displaystyle\int_\Omega |\gamma|^p \;\mathrm{d}y \leq \beta \int_\Omega W_p^{\mathrm{qc}}(\nabla u)\,\mathrm{d}y + \xi\,|\Omega|$,

*(iii)* $\displaystyle\int_\Omega \operatorname{dist}^q\big(\nabla w(\mathbb{I} - \gamma s \otimes m), \mathrm{SO}(2)\big)\,\mathrm{d}y \leq \xi\,|\Omega|$.

*PROOF of Theorem 7.28 (Upper bound inequality).* We proceed analogously to Section 7.3.2 by considering sequences $\{w_\varepsilon\}_{\varepsilon>0}$ which one gets from Lemma 7.34 by taking $\xi = \xi(\varepsilon) = \varepsilon^2$ and $\beta = \beta(\varepsilon) = 1+\varepsilon$. $\square$

### 7.4.2. More general elastic energies

So far we have been working under the assumption that our single-slip models feature an elastic energy density as simple as possible. Therefore we chose $W_{\mathrm{el}}(F) = W_{\mathrm{el},\varepsilon}(F) = \frac{1}{\varepsilon}\operatorname{dist}^q\big(F, \mathrm{SO}(2)\big)$ with $q \geq 1$ for all $F \in \mathbb{R}^{2\times 2}$ in Section 4.2. One of the drawbacks of this toy energy density is that it does not include any volumetric constraint to forbid compression of the material to zero volume by allowing for deformation gradients with non-positive determinant. In this section we want to discuss physically more realistic elastic energy densities $W_{\mathrm{el}} : \mathbb{R}^{2\times 2} \to [0,\infty]$ and to answer the question of what requirements on $W_{\mathrm{el}}$ are needed, so that the findings of Section 7.4.1 remain true. This in mind, the condensed energy density we are planning to investigate subsequently reads

$$W_{\mathrm{cond}}(F) = W_{\mathrm{cond};p}(F) = \inf_{\gamma\in\mathbb{R}} \big\{ W_{\mathrm{el}}\big(F(\mathbb{I} - \gamma s \otimes m)\big) + |\gamma|^p \big\} \tag{7.62}$$

for $F \in \mathbb{R}^{2\times 2}$ with $p \geq 1$ the plastic growth exponent.
By Chapter 2 there are two general postulates on elastic energy densities which are indispensable from the physical point of view. These are frame indifference and the property that rigid body rotations do not make any contribution to the elastic energy. Putting this into formulas leads to the hypotheses

(H1) $W_{\mathrm{el}}(RF) = W_{\mathrm{el}}(F)$ for all $R \in \mathrm{SO}(2)$ and $F \in \mathbb{R}^{2\times 2}$,

(H2) $W_{\mathrm{el}}(R) = 0$ for all $R \in \mathrm{SO}(2)$.

Let us next state the following growth condition.

(H3) There exist $N$, $M \geq 0$, $q > 0$ and constants $c_1$, $c_2 \geq 0$ such that

$$W_{\mathrm{el}}(F) \leq c_1 |F|^q + c_2$$

for all $F \in \mathbb{R}^{2\times 2}$ with $\det F > M$ and $|F| \geq N$.

Further, there is one more hypothesis mainly needed for technical reasons,

(H4) $W_{\mathrm{el}}$ is continuous at the identity.

We stress that these assumptions include energy densities $W_{\mathrm{el}}$ which are extended-valued on matrices with non-positive determinant. As we will see later on this property is widely required in applications.
Here is our first result, a further generalization of the statement of Theorem 7.26 saying that the envelopes of $W_{\mathrm{cond}}$ tend to be flat and zero in absence of hardening.

**Theorem 7.35** *Let $W_{\mathrm{cond}}$ be as in (7.62) with $p = 1$ and suppose $W_{\mathrm{el}}$ satisfies (H1)-(H4). Then, $W^{\mathrm{pc}}_{\mathrm{cond}}(F) = W^{\mathrm{rc}}_{\mathrm{cond}}(F) = W^{\mathrm{qc}}_{\mathrm{cond}}(F) = 0$ for all $F \in \mathcal{N}^{(2)}$.*

*PROOF.* The assertions concerning the polyconvex and the rank-one convex envelope are proven similarly to Theorem 7.10 and Theorem 7.26 by construction of an appropriate family of rank-one lines. For $W^{\mathrm{qc}}_{\mathrm{cond}}$, however, one needs to give a refined argument using convex integration methods. Indeed, if $W_{\mathrm{cond}}$ takes infinite values, the implication that quasiconvexity results in rank-one convexity is no longer true, see (3.5).
Since $\mathcal{N}^{(2)} = (\mathcal{M}^{(2)})^{rc}$, it is again sufficient to account only for $\mathcal{M}^{(2)}$-matrices. Then the frame indifference (H1) is used to justify the restriction to $F_* = \mathbb{I} + \sigma\gamma_0 s \otimes m$ with $\gamma_0 > 0$ and $\sigma \in \{-1, 1\}$. Following the construction in the proof of Theorem 7.10 we end up with rank-one lines $t \mapsto F_r(t)$, $t \in \mathbb{R}$ for $r > \max\{1, \gamma_0^{1/(q+1)}\}$ given by $F_r(t) = H_r \widetilde{F}_r(t)$, where $\widetilde{F}_r(t) = \mathbb{I} + tR_r$ with

$$R_r = (r-1)s \otimes s + \sigma r^{q+1} s \otimes m \tag{7.63}$$

and $H_r$ a perturbation of the identity satisfying the estimate

$$|H_r - \mathbb{I}| \leq c(\gamma_0)\, r^{-q}. \tag{7.64}$$

Then $F_r(\gamma_0 r^{-q-1}) = F_*$ and $F_r(0) = H_r$. Moreover, $F_*$ can be represented by $F_* = (1-\lambda_r)F_r(0) + \lambda_r F_r(1)$ with $\lambda_r = \gamma_0 r^{-q-1} \in (0,1)$. From hypotheses (H4), which is the continuity of $W_{\text{el}}$ in $\mathbb{I}$, (H2) and (7.64) we infer $\lim_{r\to\infty} W_{\text{el}}(F_r(0)) = W_{\text{el}}(\mathbb{I}) = 0$. Besides, with $\gamma = \sigma r^q$,

$$F_r(1)(\mathbb{I} - \gamma s \otimes m) = F_r(1)(\mathbb{I} - \sigma r^q s \otimes m) = H_r\left(rs \otimes s + m \otimes m\right). \tag{7.65}$$

Hence we have in view of (H3) that

$$W_{\text{cond}}(F_r(1)) \le c_1 |H_r|^q (r^2+1)^{q/2} + c_2 + r^q \le c(\gamma_0)\big(r^q + 1\big) \tag{7.66}$$

provided $r$ is sufficiently large, that is $r$ has to be such that $\det\left(H_r(rs \otimes s + m \otimes m)\right) > M$ and $|H_r(rs \otimes s + m \otimes m)| \ge N$. From the convexity of $W_{\text{cond}}^{\text{rc}}$ along $t \mapsto F_r(t)$ we finally infer

$$W_{\text{cond}}^{\text{rc}}(F_*) \le W_{\text{el}}\big(F_r(0)\big) + \lambda_r W_{\text{cond}}(F_r(1)) \le W_{\text{el}}\big(F_r(0)\big) + c(\gamma_0) r^{-q-1}(r^q+1) \longrightarrow 0$$

as $r \to \infty$. This proves $W^{\text{rc}}(F_*) = 0$ and as a consequence of (3.5) also $W^{\text{pc}}(F_*) = 0$.
As signalized at the beginning of this proof, a more subtle reasoning is necessary concerning the quasiconvex envelope of an extended-valued energy density $W_{\text{cond}}$. The argumentation we present here is based on the aforementioned construction. Let $\varepsilon > 0$ and for $r > \max\big\{1, \gamma_0^{1/(q+1)}\big\}$ let $\delta \in (0, \frac{1}{2}|F_r(0) - F_r(1)|) = (0, \frac{1}{2}|H_r R_r|)$ be chosen later. Then, by Lemma 5.27 there exists a function $u \in W^{1,\infty}((0,1)^2;\mathbb{R}^2)$ with $u(x) = F_* x$ for all $x \in \partial(0,1)^2$ and $\nabla u \in B(F_r(0),\delta) \cup B(F_r(1),\delta)$ almost everywhere in $(0,1)^2$. In view of Remark 3.13 we get

$$\begin{aligned} W_{\text{cond}}^{\text{qc}}(F_*) &\le \int_{(0,1)^2} W_{\text{cond}}(\nabla u)\,\mathrm{d}x \\ &= \int_{(0,1)^2 \cap \{|\nabla u - F_r(0)| < \delta\}} W_{\text{cond}}(\nabla u)\,\mathrm{d}x + \int_{(0,1)^2 \cap \{|\nabla u - F_r(1)| < \delta\}} W_{\text{cond}}(\nabla u)\,\mathrm{d}x \\ &\le \sup_{F \in B(F_r(0),\delta)} W_{\text{el}}(F) + \lambda_r \Big( \sup_{F \in B(F_r(1),\delta)} W_{\text{cond}}(F) \Big), \end{aligned} \tag{7.67}$$

recalling $\big|\{x \in (0,1)^2 \,|\, \nabla u(x) \in B(F_r(1),\delta)\}\big| = \lambda_r$ from Lemma 5.27. By an appropriate choice of $\delta > 0$ we intend to show that both terms of the right-hand side in (7.67) can be made arbitrarily small for sufficiently large $r$.
Observe that by (H4) there exists $\rho > 0$ such that $W_{\text{el}} < \varepsilon$ on $B(\mathbb{I},\rho)$. This implies

$$W_{\text{el}}(F) < \varepsilon \qquad \text{for all } \ F \in B(F_r(0), \rho/2) \tag{7.68}$$

provided $r$ is large enough to fulfill $|F_r(0) - \mathbb{I}| \le c(\gamma_0)\, r^{-q} < \rho/2$, where the first inequality is given by (7.64) .
As a next step we will show for $r$ sufficiently large that

$$W_{\text{cond}}(F) \le c\,(r^q + 1) \quad \text{ for all } \ F \in B(F_r(1), r^{-q+1}). \tag{7.69}$$

To this end suppose $F \in B(F_r(1), \mu)$ with some $\mu > 0$ (to be chosen later) and set $\hat{F} = (H_r)^{-1}F$. Exploiting that $H_r$ is close to $\mathbb{I}$ for $r$ large yields $|\hat{F} - \widetilde{F}_r(1)| \leq |H_r^{-1}|\mu < 2\mu$. Consequently there is a matrix $\hat{D} \in \mathbb{R}^{2\times 2}$ with $\hat{F} = \widetilde{F}_r(1) + \hat{D}$ and $|\hat{D}| < 2\mu$. In analogy to (7.65) we calculate

$$\begin{aligned} F(\mathbb{I} - \gamma s \otimes m) &= H_r Q \left[ \begin{pmatrix} r + D_{11} & \sigma r^{q+1} + D_{12} \\ D_{21} & D_{22} + 1 \end{pmatrix} \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix} \right] Q^T \\ &= H_r Q \begin{pmatrix} r + D_{11} & -\gamma(r + D_{11}) + \sigma r^{q+1} + D_{12} \\ D_{21} & 1 + D_{22} - \gamma D_{21} \end{pmatrix} Q^T, \end{aligned} \tag{7.70}$$

where $Q \in \mathrm{SO}(2)$ with $s = Qe_1$, $m = Qe_2$ and $D = Q^T \hat{D} Q$ with $|D| < 2\mu$. Now the parameter $\gamma$ is chosen in such a way that the (explicit) highest order term with respect to $r$ cancels, namely

$$\gamma = \frac{\sigma r^{q+1} + D_{12}}{r + D_{11}}.$$

By setting $\mu = r^{-q+1}$ one obtains $\gamma \sim r^q$ and the $(2,2)$-entry in (7.70) is at most of order one in $r$. The same is true for the other entries. In view of hypotheses (H3) we get (7.69) after all.
Finally we take $0 < \delta \leq \min\{r^{-q+1}, \rho/2, |H_r R_r|/2\}$. Then, with (7.68) and (7.69) estimate (7.67) turns into

$$W_{\mathrm{cond}}^{\mathrm{qc}}(F_*) \leq \varepsilon + c\,\lambda_r(r^q + 1) \leq \varepsilon + c\,(r^{-1} + r^{-q-1}). \tag{7.71}$$

Since (7.71) holds for all $\varepsilon > 0$ and all $r$ large enough, the claim follows immediately. □

**Remark 7.36** *Regarding the verification of Theorem 7.35 only the fact that there exists an upper polynomial bound on $W_{\mathrm{el}}$ for large matrices as in (H3) is used, while the actual value of the growth exponent plays a minor role. This is in correspondence to the result of Theorem 7.26 with $p = 1$. If we pass on to a regime including hardening, meaning $p > 1$, Theorem 7.35 can be extended to this new setting under the assumption that $q$ in (H3) additionally satisfies $q \leq \frac{p}{p-1}$.*

In the context of the Γ-convergence result of Theorem 7.18 generalizations to extended-valued elastic energies characterized by the hypotheses (H1)-(H4) are not obvious. To illustrate this let us consider for example the condensed energy density $W_{\mathrm{cond}}$ as in (7.62) with $p = 2$ and

$$W_{\mathrm{el}}(F) = \begin{cases} \frac{1}{\varepsilon}\operatorname{dist}^2(F, \mathrm{SO}(2)), & \text{if } \det F > 0, \\ \infty, & \text{else} \end{cases}$$

for $F \in \mathbb{R}^{2\times 2}$, where $\varepsilon > 0$. The results concerning compactness and the lower bound follow the established line of arguments. The main issue, though, is that all steps performed in the construction of a recovery sequence may only use matrices with positive

determinant, if the ideas of Theorem 7.18 are supposed to carry over except for minor changes. This, however, is not evident right away.
Nevertheless, Theorem 7.18 can be enhanced a little, so that we can also treat models where $W_{\mathrm{el}}$ meets (H1) and (H2) in combination with the following growth condition:

(H5) There exist constants $c_3, c_4 > 0$ such that for $F \in \mathbb{R}^{2\times 2}$,

$$c_3 \operatorname{dist}^2(F, \mathrm{SO}(2)) \leq W_{\mathrm{el}}(F) \leq c_4 \operatorname{dist}^2(F, \mathrm{SO}(2)).$$

### One specific example from applications

We want to conclude this paragraph by giving an example of a physically relevant elastic energy density which was originally proposed in [14] and further discussed in [13]. For $F \in \mathbb{R}^{2\times 2}$ let

$$W_{\mathrm{el}}^{(\kappa,\mu)}(F) = \begin{cases} U^{(\kappa,\mu)}(F) + \frac{\mu}{2}\left(|F|^2 - 2\right), & \text{if } \det F > 0, \\ \infty, & \text{else}, \end{cases} \tag{7.72}$$

where

$$U^{(\kappa,\mu)}(F) = \frac{\kappa}{4}\left((\det F)^2 - 1\right) - \frac{\kappa + 2\mu}{2} \log(\det F).$$

Here the constants $\kappa > 0$ and $\mu > 0$ stand for the bulk and elastic modulus, respectively. Further, we assume that the model at hand is free of hardening, meaning $p = 1$, and denote the corresponding condensed energy density in the sense of (7.62) by $W_{\mathrm{cond}}^{(\kappa,\mu)}$.
It is easy to check, that (H1)-(H4) are valid for (7.72), in particular the growth condition (H3) is fulfilled with $N = 0$, $M = 1$ and $q = 2$. By means of this practical example from the engineering literature we want to point out that the postulated hypotheses leave range for the inclusion of reasonable volumetric constraints. Here in fact, the elastic part of the deformation gradient encounters penalization by an increasing energy contribution the stronger the material is compressed. In formulas, $W_{\mathrm{el}}^{(\kappa,\mu)}(F) \to \infty$ as $\det F \to 0$. Besides, deformation gradients with non-positive determinant are not admissible to avoid the occurrence of zero or negative volume.
Eventually, Theorem 7.26 applies to $W_{\mathrm{cond}}^{(\kappa,\mu)}$. So for $F \in \mathcal{N}^{(2)} \setminus \mathrm{SO}(2)$ we know that $\big(W_{\mathrm{cond}}^{(\kappa,\mu)}\big)^{\mathrm{qc}}(F)$ vanishes for all $\kappa, \mu > 0$, even if there is pointwise convergence of $W_{\mathrm{cond}}^{(\kappa,\mu)}$ as $\kappa$ and $\mu$ tend to infinity to the elastically rigid $W_1$ with $W_1^{\mathrm{qc}}(F) > 0$. This qualitative difference between $W_{\mathrm{cond}}^{(\kappa,\mu)}$ and $W_1$ is rather a surprising issue, which results from the fact that we are dealing with extended-valued functions. Therefore, when calculating the relaxation of $W_{\mathrm{cond}}^{(\kappa,\mu)}$ for large $\kappa$ and $\mu$ one has to be careful not to use the analytical results for $W_1$ as an approximation.

## 7.5. Alternative approaches in a case with advantageous growth

As the proof of compactness for Theorem 7.28 shows, the generalized div-curl lemma of Section 6.5 plays the decisive role in the recovery of the incompressibility constraint, meaning in the verification of $\det \nabla u = 1$. Here we want to present two different ways of how to tackle this problem without using such a strong compensated compactness result in the setting of a single-slip model with linear hardening and fourth-order elastic energy. That is to say, in this section we study solely the functionals $E_\varepsilon = E_{\varepsilon;4,2}$ and $E = E_2$.
One of the major difficulties in the derivation of the Γ-limit for $E_\varepsilon$ is related to the anisotropic non-standard growth of the energy density $W_\varepsilon$. In the case where $p = 2$ and $q = 4$ we find in view of Lemma 7.2 that $E_\varepsilon$ is coercive with respect to $W^{1,4/3}(\Omega;\mathbb{R}^2)$. This fact will turn out advantageous for later reasoning, since the value 4/3, which characterizes the integrability of bounded energy sequences of $E_\varepsilon$, is exactly the dual exponent of $q = 4$. With this special feature argumentation becomes much easier and more elementary proofs of Theorem 7.28 are possible.

### 7.5.1. An approach via distributional determinants

The first alternative idea for achieving $\det \nabla u = 1$ is to show this property first for the distributional determinant $\operatorname{Det} \nabla u$ introduced in Section 6.2, which behaves qualitatively like $u\nabla u$.

*Alternative PROOF of Theorem 7.28 for $p = 2, q = 4$ (Compactness).* Without loss of generality we may assume here that $s = e_1$ and $m = e_2$, for the general statement is just a matter of parameter transformation. To start with let us copy all the arguments from the proof of Theorem 7.28 apart from the last step, the verification of $\det \nabla u = 1$ almost everywhere in Ω. Doing this we know that there are a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ with $E_\varepsilon[u_\varepsilon] < B$ for all $\varepsilon > 0$ and a function $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ such that

$$u_{\varepsilon_k} \rightharpoonup u \quad \text{in } W^{1,4/3}(\Omega;\mathbb{R}^2), \qquad u_{\varepsilon_k} \to u \quad \text{in } L^{4/3}(\Omega;\mathbb{R}^2), \tag{7.73}$$

and

$$\partial_1 u_{\varepsilon_k} = (\nabla u_{\varepsilon_k})e_1 \rightharpoonup (\nabla u)e_1 = \partial_1 u \quad \text{in } L^4(\Omega;\mathbb{R}^2) \tag{7.74}$$

as $k \to \infty$. Hence, Proposition 6.5 applied with $q = 4$ yields

$$\|u_{\varepsilon_k}\|_{L^8(\Omega;\mathbb{R}^2)} \le c\left(\|u_{\varepsilon_k}\|_{W^{1,4/3}(\Omega;\mathbb{R}^2)} + \|\partial_1 u_{\varepsilon_k}\|_{L^4(\Omega;\mathbb{R}^2)}\right) < C < \infty \tag{7.75}$$

for all $k \in \mathbb{N}$. In view of (7.73) and (7.75), Lemma 6.7 with $p = 8$ and $q = 4$ gives $u_{\varepsilon_k} \to u$ in $L^4(\Omega;\mathbb{R}^2)$, in particular $u_{\varepsilon_k} \cdot e_2 \to u \cdot e_2$ in $L^4(\Omega)$. With formulation (6.14) one can now pass to the limit in the distributional determinant of $\nabla u_{\varepsilon_k}$ by exploiting the strong $L^4$-convergence of $u_{\varepsilon_k} \cdot e_2$ together with $\nabla(u_{\varepsilon_k} \cdot e_1) \rightharpoonup \nabla(u \cdot e_1)$ in $L^{4/3}(\Omega;\mathbb{R}^2)$. Thus,

$$\begin{aligned} \lim_{k\to\infty} \langle \mathrm{Det}\, \nabla u_{\varepsilon_k}, \varphi\rangle &= \lim_{k\to\infty} \int_\Omega (u_{\varepsilon_k} \cdot e_2)\big(\nabla(u_{\varepsilon_k} \cdot e_1) \cdot J\nabla\varphi\big)\, \mathrm{d}x \\ &= \int_\Omega (u \cdot e_2)\big(\nabla(u \cdot e_1) \cdot J\nabla\varphi\big)\, \mathrm{d}x = \langle \mathrm{Det}\, \nabla u, \varphi\rangle \end{aligned} \tag{7.76}$$

for all $\varphi \in C_0^\infty(\Omega)$. As a consequence of Lemma 7.4 it holds $\det \nabla u_{\varepsilon_k} \to 1$ in $L^1(\Omega)$ for $k \to \infty$. On the other hand it has just been shown in (7.76) that $\mathrm{Det}\, \nabla u_{\varepsilon_k} \to \mathrm{Det}\, \nabla u$ in $\mathcal{D}'(\Omega)$. In order to join these two results together we make use of Lemma 6.6, which in view of (7.73) and (7.74) leads to $\mathrm{Det}\, \nabla u_{\varepsilon_k} = \det \nabla u_{\varepsilon_k}$ in $\mathcal{D}'(\Omega)$ for all $k \in \mathbb{N}$, and get

$$\begin{aligned} \int_\Omega \det \nabla u\, \varphi\, \mathrm{d}x &= \langle \mathrm{Det}\, \nabla u, \varphi\rangle = \lim_{k\to\infty} \langle \mathrm{Det}\, \nabla u_{\varepsilon_k}, \varphi\rangle \\ &= \lim_{k\to\infty} \int_\Omega \det \nabla u_{\varepsilon_k} \varphi\, \mathrm{d}x = \int_\Omega \varphi\, \mathrm{d}x \end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$. Notice that the first equality is due to the fact that $\det \nabla u$ and $\mathrm{Det}\, \nabla u$ coincide for $u \in W^{1,2}(\Omega;\mathbb{R}^2)$. Finally, from the fundamental theorem of the calculus of variations one infers $\det \nabla u = 1$ almost everywhere in $\Omega$. □

### 7.5.2. An approach using lower semicontinuity results

The next alternative proof for the liminf-inequality and the recovery of the incompressibility in the Γ-convergence result of Theorem 7.28 with $p = 2$ and $q = 4$ is based on a lower semicontinuity result for polyconvex functions, which is an appropriate modification of [36, Theorem 3.3] to the needs of the situation at hand. In order to better compare our version to the one by Fusco and Hutchinson we briefly state their setting and quote [36, Theorem 3.3] right below. Recall that the vector of all minors of a matrix $F \in \mathbb{R}^{m\times n}$ is denoted by $M(F)$. In the 2D case this is $M(F) = (F, \det F)$ for $F \in \mathbb{R}^{2\times 2}$.

**Theorem 7.37** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $f : \mathbb{R}^{m\times n} \to \mathbb{R}_0^+$ be polyconvex and $l = \min\{n, m\}$. If $\{u^k\}_{k\in\mathbb{N}} \subset W^{1,l}(\Omega;\mathbb{R}^m)$ and $u \in W^{1,1}(\Omega;\mathbb{R}^m)$ such that $u^k \to u$ in $L^1(\Omega;\mathbb{R}^m)$ as $k \to \infty$ and*

$$\sup_{k\in\mathbb{N}} \int_\Omega \big|M(\nabla u^k)\big|\, \mathrm{d}x < \infty,$$

*then $\int_\Omega f(\nabla u)\, \mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega f(\nabla u^k)\, \mathrm{d}x$.*

**Remark 7.38** *Actually, in [36] the function $f$ is even more general inasmuch as it may depend on $x$ and $u$, i.e. $f(x,y,F) : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}_0^+$. Then, additionally to the polyconvexity of $f$ in $F$ the following type of uniform lower bound has to be postulated:*

*For any $(x_0, y_0) \in \Omega \times \mathbb{R}^m$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $f(x,y,F) \geq (1-\varepsilon) f(x_0, y_0, F)$, if $|x - x_0| < \delta$, $|y - y_0| < \delta$ and $F \in \mathbb{R}^{m\times n}$.*

*Since our energy densities $W_\varepsilon$ neither depend on $x$ nor on $u$ this condition is always true. Therefore we skipped the explicit dependence on $x$ and $u$.*

The reason why the lower semicontinuity result of Theorem 7.37 is not appropriate for our purpose is that the sequence $\{u^k\}_{k\in\mathbb{N}}$, which corresponds to a bounded energy sequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $E_{\varepsilon_k}$ in our application in mind, is required to consist of $W^{1,2}$-functions in the two-dimensional setting with $n = m = 2$. However, the growth conditions of $W_{\varepsilon_k}$ merely give $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}} \subset W^{1,4/3}(\Omega;\mathbb{R}^2)$. With this motivation we formulate the following theorem.

**Theorem 7.39** *Let $\Omega \subset \mathbb{R}^2$ be bounded domain, $p \in (1,2)$ and $q = \max\{\frac{2p}{2-p}, \frac{p}{p-1}\}$. Further let $f : \mathbb{R}^{2\times 2} \to \mathbb{R}_0^+$ be a polyconvex function. Suppose $\{u^k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^2)$ and $u \in W^{1,1}(\Omega;\mathbb{R}^2)$ are such that $u^k \to u$ in $L^1(\Omega;\mathbb{R}^2)$ as $k \to \infty$ and that there exists a constant $C < \infty$ with*

$$\|u^k\|_{W^{1,p}(\Omega;\mathbb{R}^2)} \leq C \quad \text{and} \quad \|(\nabla u^k)e_1\|_{L^q(\Omega;\mathbb{R}^2)} = \|\partial_1 u^k\|_{L^q(\Omega;\mathbb{R}^2)} \leq C \tag{7.77}$$

*for all $k \in \mathbb{N}$. Then $\int_\Omega f(\nabla u)\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega f(\nabla u^k)\,\mathrm{d}x$.*

**Remark 7.40** *Note that requirement (7.77) implies the existence of a uniform bound on $\int_\Omega |M(\nabla u^k)|\,\mathrm{d}x$. Indeed, by Hölder's inequality,*

$$\int_\Omega |\det \nabla u^k|\,\mathrm{d}x \leq \int_\Omega |\partial_1 u^k|\,|\partial_2 u^k|\,\mathrm{d}x \leq c(p,\Omega)\,\|\partial_1 u^k\|_{L^q(\Omega;\mathbb{R}^2)}\|u^k\|_{W^{1,p}(\Omega;\mathbb{R}^2)} \leq C < \infty.$$

Let us state two general results from the literature and show an auxiliary proposition, before we focus on the proof Theorem 7.39. The first result we want to quote is [36, Proposition 3.1], the proof of which follows closely the lines of [34, Theorem 3.3].

**Proposition 7.41** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Suppose $g : \mathbb{R}^{m\times n} \to \mathbb{R}_0^+$ is a polyconvex function with $|M(F)| \leq g(F) \leq c\,(1 + |M(F)|)$ for all $F \in \mathbb{R}^{m\times n}$ and a constant $c > 0$. If $\{u^k\}_{k\in\mathbb{N}} \subset C^1(\Omega;\mathbb{R}^m)$ and $u \in W^{1,1}(\Omega;\mathbb{R}^m)$ with $u^k \to u$ in $L^1(\Omega;\mathbb{R}^m)$, then*

$$\liminf_{k\to\infty} \int_\Omega g(\nabla u^k)\,\mathrm{d}x \geq \int_\Omega g(\nabla u)\,\mathrm{d}x.$$

Next we give an approximation result for polyconvex functions which is again taken from [36] and is a helpful tool for carrying the statement of Proposition 7.41 over to general polyconvex functions.

**Lemma 7.42 ([36, Lemma 3.2])** *Suppose $f : \mathbb{R}^{m\times n} \to \mathbb{R}_0^+$ is polyconvex. Then there exists a sequence of continuous, polyconvex functions $\{g_j\}_{j\in\mathbb{N}}$ mapping from $\mathbb{R}^{m\times n}$ into $\mathbb{R}_0^+$ such that $0 \le g_j(F) \le c_j\,(1 + |M(F)|)$ with a constant $c_j > 0$ and $g_j(F) \le g_{j+1}(F)$ for all $j \in \mathbb{N}$, $F \in \mathbb{R}^{m\times n}$ and $f(F) = \sup_{j\in\mathbb{N}} g_j(F)$ for all $F \in \mathbb{R}^{m\times n}$.*

Here is another approximation tool needed in the proof of Theorem 7.39.

**Proposition 7.43** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $p \in (1, 2)$ and $q = \max\{\frac{2p}{2-p}, \frac{p}{p-1}\}$. Moreover, suppose the functional $G : W^{1,1}(\Omega;\mathbb{R}^2) \to \mathbb{R}_0^+$ is defined by*

$$G[u] = \int_\Omega g(\nabla u)\,\mathrm{d}x,$$

*where $g : \mathbb{R}^{2\times2} \to \mathbb{R}_0^+$ is polyconvex satisfying $0 \le g(F) \le c\,(1 + |M(F)|)$ for all $F \in \mathbb{R}^{2\times2}$ with a constant $c > 0$. Then, for every $\delta > 0$ and every $u \in W^{1,p}(\Omega;\mathbb{R}^2)$ with $\partial_1 u \in L^q(\Omega;\mathbb{R}^2)$ there exists $v \in C^\infty(\Omega;\mathbb{R}^2)$ such that*

*(i)* $\|u - v\|_{W^{1,p}(\Omega;\mathbb{R}^2)} \le \delta$,

*(ii)* $\|(\nabla v)e_1 - (\nabla u)e_1\|_{L^q(\Omega;\mathbb{R}^2)} \le \delta$,

*(iii)* $G(v) \le G(u) + \delta$.

*PROOF.* We concentrate on (iii), for (i) and (ii) are an immediate consequence of Lemma 6.4. Since $g$ is polyconvex, there is a convex function $\bar g : \mathbb{R}^5 \to \mathbb{R}_0^+$ such that $g(F) = \bar g(F, \det F)$. By assumption $\bar g$ fulfills the growth condition

$$0 \le \bar g(F, \det F) \le c\,(1 + |F| + |\det F|) \tag{7.78}$$

for all $F \in \mathbb{R}^{2\times2}$, so that $\bar g$ is proper with $\operatorname{dom}\bar g = \{x \in \mathbb{R}^5 \,|\, x = (F, \det F),\ F \in \mathbb{R}^{2\times2} \cong \mathbb{R}^4\}$. Here $\operatorname{dom}\bar g$ stands for the set of points where $\bar g$ is finite. Then a standard result from convex analysis, see i.e. [69, Theorem 23.4], implies

$$\partial\bar g(F, \det F) \ne \emptyset \tag{7.79}$$

for all $F \in \mathbb{R}^{2\times2}$, where $\partial\bar g$ denotes the subdifferential of the proper and convex function $\bar g$. Let us prove next that growth condition (7.78) yields a uniform upper bound for all subgradients $(P, p) \in \partial\bar g(F, \det F)$ with $F \in \mathbb{R}^{2\times2}$ where $P = (P_{ij})_{i,j=1,2} \in \mathbb{R}^{2\times2} \cong \mathbb{R}^4$. Using the definition of the subdifferential together with (7.78) one obtains

$$\begin{aligned} c\,(1 + |H| + |\det H|) &\ge \bar g(H, \det H) \\ &\ge \bar g(F, \det F) + P : (H - F) + p\,(\det H - \det F) \end{aligned} \tag{7.80}$$

for all $H \in \mathbb{R}^{2\times 2}$. In particular, (7.80) holds true for all matrices of the form $H = (\operatorname{sign} P_{ij})\lambda e_i \otimes e_j$ with $i, j \in \{1,2\}$ and $\lambda > 0$ so that

$$c\,(1+|\lambda|) \geq \bar{g}(F, \det F) - P : F - p \det F + \lambda |P_{ij}|.$$

Considering the limit $\lambda \to \infty$ provides $|P_{ij}| \leq c$ for all $i, j \in \{1,2\}$ and consequently

$$|P| \leq \sum_{i=1}^{2}\sum_{j=1}^{2} |P_{ij}| \leq 4\,c. \tag{7.81}$$

With the second choice $H = \lambda\, e_1 \otimes e_1 + (\operatorname{sign} p)\lambda\, e_2 \otimes e_2$ for $\lambda > 0$ we find $c\,(1+\sqrt{2}\lambda+\lambda^2) \geq \bar{g}(F, \det F) - P : F - p : \det F - \sqrt{2}\lambda\,|P| + \lambda^2 |p|$. Taking the limit $\lambda \to \infty$ results in

$$|p| \leq c. \tag{7.82}$$

Let $\mu > 0$ (to be chosen later) and take $u$ as in the statement of Proposition (7.43). Further let $v \in C^\infty(\Omega;\mathbb{R}^2)$ be the function resulting from Lemma 6.4, so that $\|u - v\|_{W^{1,p}(\Omega;\mathbb{R}^m)} \leq \mu$, and $\|\partial_1 u - \partial_1 v\|_{L^q(\Omega;\mathbb{R}^m)} \leq \mu$. Then, by the definition of the subdifferential, Hölder's inequality and the bounds (7.81) and (7.82),

$$\begin{aligned}
G(v) - G(u) &= \int_\Omega \bar{g}(\nabla v, \det \nabla v) - \bar{g}(\nabla u, \det \nabla u)\,\mathrm{d}x \\
&\leq \int_\Omega P : (\nabla v - \nabla u) + p\,(\det \nabla v - \det \nabla u)\,\mathrm{d}x \\
&\leq 4\,c\,\|\nabla v - \nabla u\|_{L^1(\Omega;\mathbb{R}^{2\times 2})} + c\,\|\det \nabla v - \det \nabla u\|_{L^1(\Omega)} \qquad (7.83)\\
&\leq c(p,\Omega)\Big(|\Omega|^{(p-1)/p}\|v-u\|_{W^{1,p}(\Omega;\mathbb{R}^2)} + \|\partial_1 u\|_{L^q(\Omega;\mathbb{R}^2)}\|u-v\|_{W^{1,p}(\Omega;\mathbb{R}^2)} \\
&\qquad\qquad + \|\partial_1 v - \partial_1 u\|_{L^q(\Omega;\mathbb{R}^2)}\|v\|_{W^{1,p}(\Omega;\mathbb{R}^2)}\Big) \\
&\leq c(p,\Omega)\,\mu^2 + c(p,\Omega)\,\mu\left(|\Omega|^{(p-1)/p} + \|\partial_1 u\|_{L^q(\Omega;\mathbb{R}^2)} + \|u\|_{W^{1,p}(\Omega;\mathbb{R}^2)}\right),
\end{aligned}$$

where $P : \Omega \to \mathbb{R}^{2\times 2}$ and $p : \Omega \to \mathbb{R}$ are such that $\big(P(x), p(x)\big) \in \partial\bar{g}\big(\nabla v(x), \det \nabla v(x)\big)$ for almost all $x \in \Omega$. Finally the claim follows, if $\mu$ is chosen small enough so that the right-hand side in (7.83) is smaller than $\delta$. □

*PROOF of Theorem 7.39:* For the polyconvex $f : \mathbb{R}^{2\times 2} \to \mathbb{R}_0^+$ we choose a sequence of continuous, polyconvex functions $\{g_j\}_{j\in\mathbb{N}}$ according to Lemma 7.42 and define functionals

$$G_j : W^{1,1}(\Omega;\mathbb{R}^2) \to \mathbb{R}_0^+, \quad u \mapsto \int_\Omega g_j(\nabla u)\,\mathrm{d}x \tag{7.84}$$

for $j \in \mathbb{N}$. Let $\{u^k\}_{k\in\mathbb{N}}$ and $u$ be as in the statement of Theorem 7.39. By Proposition 7.43 there exists for every $k \in \mathbb{N}$ and $j \in \mathbb{N}$ a function $v_j^k \in C^\infty(\Omega;\mathbb{R}^2)$ with

$$\begin{gathered}
\|u^k - v_j^k\|_{W^{1,p}(\Omega;\mathbb{R}^2)} \leq 1/k, \qquad \|(\nabla v_j^k)e_1 - (\nabla u^k)e_1\|_{L^q(\Omega;\mathbb{R}^2)} \leq 1/k \\
\text{and} \;\; G_j(v_j^k) \leq G_j(u^k) + 1/k.
\end{gathered} \tag{7.85}$$

In view of the first of these three properties and the assumptions on $\{u^k\}_{k\in\mathbb{N}}$ one finds $v_j^k \to u$ in $L^1(\Omega;\mathbb{R}^2)$ as $k\to\infty$ for all $j\in\mathbb{N}$. Moreover there is a uniform $L^1$-bound on the minors of $\nabla v_j^k$. Indeed,

$$\begin{aligned}
\sup_{k,j\in\mathbb{N}}\int_\Omega \big|M(\nabla v_j^k)\big|\,\mathrm{d}x &\le \sup_{k,j\in\mathbb{N}}\int_\Omega |\nabla v_j^k|\,\mathrm{d}x + \sup_{k,j\in\mathbb{N}}\int_\Omega |\det\nabla v_j^k|\,\mathrm{d}x \qquad (7.86)\\
&\le |\Omega|^{(p-1)/p}\sup_{k,j\in\mathbb{N}}\|v_j^k\|_{W^{1,p}(\Omega;\mathbb{R}^2)} + c(p,\Omega)\sup_{k,j\in\mathbb{N}}\|(\nabla v_j^k)e_1\|_{L^q(\Omega;\mathbb{R}^2)}\|v_j^k\|_{W^{1,p}(\Omega;\mathbb{R}^2)}\\
&\le c(p,\Omega)\Bigg[\sup_{k\in\mathbb{N}}\big(\|u^k\|_{W^{1,p}(\Omega;\mathbb{R}^2)}+1/k\big)\\
&\qquad + \sup_{k\in\mathbb{N}}\Big(\big(\|(\nabla u^k)e_1\|_{L^q(\Omega;\mathbb{R}^2)}+1/k\big)\big(\|u^k\|_{W^{1,p}(\Omega;\mathbb{R}^2)}+1/k\big)\Big)\Bigg] < C < \infty,
\end{aligned}$$

since $\|u^k\|_{W^{1,p}(\Omega;\mathbb{R}^2)}$ and $\|(\nabla u^k)e_1\|_{L^q(\Omega;\mathbb{R}^2)}$ are uniformly bounded by assumption. With these preliminaries at hand the idea is to apply Proposition 7.41 to suitable modifications of the functions $g_j$, namely to

$$g_{j,\varepsilon}:\mathbb{R}^{2\times 2}\to\mathbb{R}_0^+,\quad F\mapsto g_{j,\varepsilon}(F)=\frac{1}{\varepsilon}\Big(g_j(F)+\varepsilon|M(F)|\Big)$$

with $\varepsilon>0$. We observe that the polyconvexity and positivity of $g_j$ carry immediately over to $g_{j,\varepsilon}$ and we have the growth condition

$$|M(F)|\le g_{j,\varepsilon}(F)\le\frac{1}{\varepsilon}\big(c_j(1+|M(F)|)\big)+|M(F)|\le c_{j,\varepsilon}\big(1+|M(F)|\big)$$

for all $F\in\mathbb{R}^{2\times 2}$, where $c_{j,\varepsilon}=1+\frac{c_j}{\varepsilon}$. Then, for every $j\in\mathbb{N}$ and every $\varepsilon>0$ the requirements of Proposition 7.41 are fulfilled by $g_{j,\varepsilon}$ and $\{v_j^k\}_{k\in\mathbb{N}}$, so that

$$\int_\Omega g_{j,\varepsilon}(\nabla u)\,\mathrm{d}x \le \liminf_{k\to\infty}\int_\Omega g_{j,\varepsilon}(\nabla v_j^k)\,\mathrm{d}x. \qquad (7.87)$$

From (7.87), (7.86), (7.85) and the fact that $g_j\le f$ for all $j\in\mathbb{N}$ by Lemma 7.42 one infers

$$\begin{aligned}
\int_\Omega g_j(\nabla u)\,\mathrm{d}x &\le \int_\Omega \varepsilon\, g_{j,\varepsilon}(\nabla u)\,\mathrm{d}x \le \liminf_{k\to\infty}\int_\Omega \varepsilon\, g_{j,\varepsilon}(\nabla v_j^k)\,\mathrm{d}x\\
&= \liminf_{k\to\infty}\int_\Omega g_j(\nabla v_j^k)+\varepsilon|M(\nabla v_j^k)|\,\mathrm{d}x\\
&\le \liminf_{k\to\infty}\left(\int_\Omega g_j(\nabla u^k)\,\mathrm{d}x + 1/k\right) + \varepsilon\sup_{k,j\in\mathbb{N}}\int_\Omega |M(\nabla v_j^k)|\,\mathrm{d}x\\
&\le \liminf_{k\to\infty}\int_\Omega f(\nabla u^k)\,\mathrm{d}x + \varepsilon\, C
\end{aligned}$$

for all $\varepsilon > 0$ and $j \in \mathbb{N}$. Since $\varepsilon > 0$ is arbitrary in the above calculation, we deduce $\int_\Omega g_j(\nabla u)\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega f(\nabla u^k)\,\mathrm{d}x$ for all $j \in \mathbb{N}$. Finally, in view of the monotonicity of $\{g_j\}_{j\in\mathbb{N}}$ by Lemma 7.42 one infers from Lebesgue's monotone convergence theorem that

$$\int_\Omega f(\nabla u)\,\mathrm{d}x = \lim_{j\to\infty} \int_\Omega g_j(\nabla u)\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega f(\nabla u^k)\,\mathrm{d}x,$$

which is the asserted statement. □

*Alternative PROOF of Theorem 7.28 for $p = 2$, $q = 4$ (Compactness and lower bound inequality).* We may assume here that $s = e_1$ and $m = e_2$. Then the general statement is just a matter of parameter transformation. As in the proof of Theorem 7.28 given in Section 7.4.1 we show that a finite energy sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of $E_\varepsilon$ satisfies the uniform bounds

$$\|u_\varepsilon\|_{W^{1,4/3}(\Omega;\mathbb{R}^2)} \leq C \quad \text{and} \quad \|(\nabla u_\varepsilon)e_1\|_{L^4(\Omega;\mathbb{R}^2)} \leq C$$

and has a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ such that $u_{\varepsilon_k} \to u$ in $L^{4/3}(\Omega;\mathbb{R}^2)$ with $u \in W^{1,4/3}(\Omega;\mathbb{R}^2)$. Recall that $W_\varepsilon$ is monotone with respect to $\varepsilon$ by Lemma 7.5. Then, for any $\eta > 0$ we infer from Theorem 7.39 applied with $f = W_\eta^{\mathrm{pc}}$, $\{u^k\}_{k\in\mathbb{N}} = \{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$, $p = 4/3$ and $q = 4$ that

$$\begin{aligned}
\liminf_{k\to\infty} \int_\Omega W_{\varepsilon_k}(\nabla u_{\varepsilon_k})\,\mathrm{d}x &\geq \liminf_{k\to\infty} \int_\Omega W_\eta(\nabla u_{\varepsilon_k})\,\mathrm{d}x \\
&\geq \liminf_{k\to\infty} \int_\Omega W_\eta^{\mathrm{pc}}(\nabla u_{\varepsilon_k})\,\mathrm{d}x \geq \int_\Omega W_\eta^{\mathrm{pc}}(\nabla u)\,\mathrm{d}x.
\end{aligned}$$

By the pointwise convergence $\lim_{\eta\to 0} W_\eta^{\mathrm{pc}}(F) = W^{\mathrm{pc}}(F)$ for all $F \in \mathbb{R}^{2\times 2}$ of Theorem 7.27 we find

$$\begin{aligned}
\liminf_{k\to\infty} \int_\Omega W_{\varepsilon_k}(\nabla u_{\varepsilon_k})\,\mathrm{d}x &\geq \liminf_{\eta\to 0} \int_\Omega W_\eta^{\mathrm{pc}}(\nabla u)\,\mathrm{d}x \geq \int_\Omega \lim_{\eta\to 0} W_\eta^{\mathrm{pc}}(\nabla u)\,\mathrm{d}x \\
&= \int_\Omega W^{\mathrm{pc}}(\nabla u)\,\mathrm{d}x = \int_\Omega W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x. \qquad (7.88)
\end{aligned}$$

For the second inequality we applied Fatou's lemma, which is possible because the polyconvex energy densities $W_\eta^{\mathrm{pc}}$ are nonnegative. The last step is due to Theorem 4.1, where equality of the polyconvex and quasiconvex envelope of $W$ was shown. This finishes the proof of the lower bound and at the same time provides information about the limit function $u$. Indeed, in view of (7.88) we see $\int_\Omega W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x < \infty$. Hence, $\nabla u \in \mathcal{N}^{(2)}$ almost everywhere in $\Omega$ and $u \in W^{1,2}(\Omega;\mathbb{R}^2)$. □

# 8. Results for the three-dimensional setting

Our intention for this chapter is to study the single-slip models of Section 4.2 in the setting of three space dimensions and to compare the findings to the 2D situation discussed in Chapter 7. From now on we assume for the slip direction $s$ that $s = e_1$ and take $m = e_2$ as a slip plane normal. Then the condensed energy density we want to investigate in the following is

$$W_\varepsilon(F) = W_{\varepsilon;q,p}(F) = \inf_{\gamma\in\mathbb{R}} \Big\{ \frac{1}{\varepsilon} \ \mathrm{dist}^q \big(F(\mathbb{I} - \gamma e_1 \otimes e_2), \mathrm{SO}(3)\big) + |\gamma|^p \Big\}$$

for all $F \in \mathbb{R}^{3\times 3}$ with growth exponents $p, q \in [1, \infty)$.
For the readers' convenience we repeat here in short the most important notions and the relevant results from Chapter 4. Indeed, in 3D the situation is so rigid that $W = W_p$ coincides with its quasiconvex envelope according to Theorem 4.1, while the rank-one and polyconvex envelopes are given by

$$W_p^{\mathrm{rc}}(F) = W_p^{\mathrm{pc}}(F) = \begin{cases} \big(|Fe_2|^2 - 1\big)^{p/2} & \text{for } F \in \mathcal{N}^{(3)}, \\ \infty & \text{else,} \end{cases}$$

if $p \geq 2$ and for $p = 1$ by

$$W_1^{\mathrm{rc}}(F) = W_1^{\mathrm{pc}}(F) = \begin{cases} \sqrt{|F|^2 - 3} & \text{for } F \in \mathcal{N}^{(3)}, \\ \infty & \text{else.} \end{cases}$$

Let us recall that the set $\mathcal{N}^{(3)}$ can be written in various ways. That is

$$\begin{aligned} \mathcal{N}^{(3)} &= \big\{F \in \mathbb{R}^{3\times 3} |\ \det F = 1,\ |(\mathrm{cof}\, F)e_3| = |Fe_3| = 1,\ |Fe_1| \leq 1\big\} \\ &= \big\{F \in \mathbb{R}^{3\times 3} |\ \det F = 1,\ (\mathrm{cof}\, F)e_3 = Fe_3,\ |Fe_1| \leq 1\big\} = (\mathcal{M}^{(3)})^{\mathrm{rc}}. \end{aligned}$$

Here we follow the same outline as in Chapter 7 and start by proving soft material behavior for the model without hardening. Subsequently, we include hardening with its regularizing effect. This will enable us to give a $\Gamma$-convergence result in the sense of Theorem 7.28 and to make a statement on the asymptotic behavior of the single-slip systems in the limit of rigid elasticity. In contrast to the $\Gamma$-limit in two space dimensions, however, the 3D $\Gamma$-limit is characterized by the rank-one convex envelope of $W$ instead of its relaxation $W^{\mathrm{qc}}$. In order to achieve Theorem 8.3 along the lines of Section 7.3.2 technical reasons require higher elastic and plastic growth exponents than in 2D. In particular, one needs to avoid working in $L^p$-spaces with $0 < p < 1$.

## 8.1. Relaxation of the model without hardening

Next we present the three-dimensional analogue of Theorem 7.10, meaning that we investigate $W_\varepsilon = W_{\varepsilon;2,1}$. It is helpful to observe that this problem has substantially only two space dimensions.

**Theorem 8.1** *For $\varepsilon > 0$ it holds $W_\varepsilon^{\mathrm{pc}}(F) = W_\varepsilon^{\mathrm{qc}}(F) = W_\varepsilon^{\mathrm{rc}}(F) = 0$ for all $F \in \mathcal{N}^{(3)}$.*

*PROOF.* Since $\mathcal{N}^{(3)} = (\mathcal{M}^{(3)})^{\mathrm{rc}}$, it suffices to restrict attention to $F \in \mathcal{M}^{(3)}$. By the frame indifference of $W_\varepsilon$ this means that we need to focus on matrices of the form

$$F_{**} = \begin{pmatrix} F_* & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}^{(3)}, \quad \text{where } F_* = \begin{pmatrix} 1 & \sigma\gamma_0 \\ 0 & 1 \end{pmatrix} = \mathbb{I} + \sigma\gamma_0\, e_1 \otimes e_2 \ \in \mathcal{M}^{(2)}$$

with $\gamma_0 > 0$ and $\sigma \in \{-1, 1\}$. The matrix $F_{**} \in \mathbb{R}^{3\times 3}$ being simply the trivial extension of $F_* \in \mathbb{R}^{2\times 2}$ the problem is in fact two-dimensional. This allows us to disregard the additional space dimension and to copy the construction for the proof of Theorem 7.10. □

**Remark 8.2** *Note that enhancements of this result to more general elastic energies and growth exponents in the sense of Section 7.4 are possible as well. Just extend the constructions proving Theorem 7.26 and 7.35 trivially to one more dimension.*

## 8.2. Asymptotic behavior in 3D

The remarkable finding in contrast to the 2D setting of Section 7.4.1 is that here in three dimensions $W^{\mathrm{qc}}$ is too rigid to characterize the desired Γ-limit of $E_\varepsilon$. Instead, a term involving the rank-one convex envelope of the elastically rigid energy density turns out to give the correct expression. As before, the particular difficulty in proving Γ-convergence in Theorem 8.3 lies in the constraint of incompressibility, which requires good growth conditions of the elastic and plastic energy and a subtle compensated compactness result for its recovery in the limit.

### 8.2.1. Formulation of the main theorem

Let the elastic and plastic growth exponents $p, q \in [1, \infty)$ satisfy the postulates

(P1) $p \geq 2$, (P2) $q \geq 3$,

(P3) $q \geq \frac{2p}{p-1}$, (P4) $q \leq \frac{3}{2}p$.

For an illustration of the relation between $p$ and $q$ see Figure 8.1.
Now we are in the position to formulate the main theorem of this paragraph.

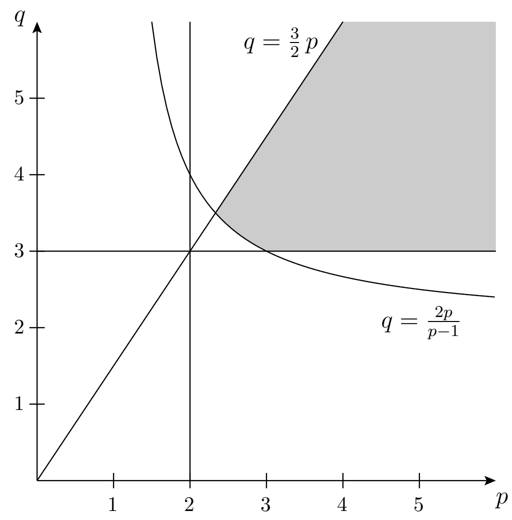


Figure 8.1.: Illustration of (P1) - (P4)

**Theorem 8.3** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $p, q$ satisfy (P1)-(P4). Then the functionals $E_\varepsilon = E_{\varepsilon;q,p}$ with $\varepsilon > 0$ defined as*

$$E_{\varepsilon;q,p}[u] = \int_\Omega W_{\varepsilon;q,p}(\nabla u)\,\mathrm{d}x$$

*converge in the sense of $\Gamma$-convergence with respect to strong convergence in $L^{\frac{pq}{p+q}}$ to the variational integral $E = E_p$ given by*

$$E_p[u] = \begin{cases} \displaystyle\int_\Omega W_p^{\mathrm{rc}}(\nabla u)\,\mathrm{d}x, & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^3),\ \nabla u \in \mathcal{N}^{(3)} \text{ a.e. in } \Omega\,, \\ \infty, & \text{else,} \end{cases}$$

*as $\varepsilon$ tends to zero. Moreover, any bounded energy sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of $E_\varepsilon$ is relatively compact in $L^{\frac{pq}{p+q}}$.*

**Remark 8.4** *The hypotheses (P2) and (P3) are imposed to guarantee the necessary control on determinants of gradients of low energy sequences $\{u_\varepsilon\}_{\varepsilon>0}$ for $E_\varepsilon$ required in the proof of compactness. More precisely, in view of the algebraic estimate Lemma 8.5 (v) postulate (P2) implies that* $\det \nabla u_\varepsilon$ *is integrable for all $\varepsilon > 0$. With the help of (P3), which is equivalent to $\frac{pq}{2p+q} \geq 1$, one can show the passage of the determinant constraint to the limit, see* (8.3). *Constraint (P1) results from the disposability of rank-one convex envelopes for $W_p$ by Theorem 4.1 in combination with the result of Theorem 8.1. Besides, (P4) is due to the natural limits of the method relying on Sobolev embeddings which was used for the construction of the recovery sequence in Lemma 8.9.*

For the proof of Theorem 8.3 the by now three well-known steps of showing compactness and the lower bound inequality and of constructing an appropriate recovery sequence have to be performed. Many of the arguments are analogous to what was done in the proof of Theorem 7.28. With respect to the liminf inequality they are even the same, which is why we dispense with writing them down once again.

### 8.2.2. Proofs

In analogy to (7.4) we obtain an advantageous representation of $W_\varepsilon$, if $q \geq 2$. Note that every $Q \in \mathrm{SO}(3)$ can be expressed as $Q = a \otimes e_1 + b \otimes e_2 + (a \wedge b) \otimes e_3$ with $a, b \in \mathbb{S}^2$ and $a \perp b$. Hence, for $F \in \mathbb{R}^{3\times 3}$ one may define $a_\varepsilon(F) \in \mathbb{S}^2$, $b_\varepsilon(F) \in \mathbb{S}^2$ and $\gamma_\varepsilon(F) \in \mathbb{R}$ by

$$\begin{aligned}
W_\varepsilon(F) &= W_{\varepsilon;q,p}(F) \\
&= \min_{\gamma\in\mathbb{R},\, a,b\in\mathbb{S}^2, a\perp b} \left\{ \frac{1}{\varepsilon} \left( |Fe_1 - a|^2 + |Fe_2 - \gamma F e_1 - b|^2 + |Fe_3 - (a\wedge b)|^2 \right)^{q/2} + |\gamma|^p \right\} \\
&\geq \min_{\gamma\in\mathbb{R},\, a,b\in\mathbb{S}^2, a\perp b} \left\{ \frac{1}{\varepsilon} \left( |Fe_1 - a|^q + |Fe_2 - Fe_1 - b|^q + |Fe_3 - (a\wedge b)|^q \right) + |\gamma|^p \right\} \\
&= \frac{1}{\varepsilon} \big( |Fe_1 - a_\varepsilon(F)|^q + |Fe_2 - \gamma_\varepsilon(F) F e_1 - b_\varepsilon(F)|^q \\
&\qquad\qquad + |Fe_3 - (a_\varepsilon(F) \wedge b_\varepsilon(F))|^q \big) + |\gamma_\varepsilon(F)|^p .
\end{aligned}$$

The inequality follows from (7.5) by setting $y = \left(|Fe_1 - a|^q + |Fe_2 - \gamma F e_1 - b|^q\right)^{1/q}$ and $z = |Fe_3 - (a \wedge b)|$. As in Section 7.1 we suppress the argument $F$ and write $a_\varepsilon$, $b_\varepsilon$ and $\gamma_\varepsilon$ instead of $a_\varepsilon(F)$, $b_\varepsilon(F)$ and $\gamma_\varepsilon(F)$, respectively. Further we denote $c_\varepsilon = a_\varepsilon \wedge b_\varepsilon$, so that finally

$$W_\varepsilon(F) \geq \frac{1}{\varepsilon} \left( |Fe_1 - a_\varepsilon|^q + |Fe_2 - \gamma_\varepsilon F e_1 - b_\varepsilon|^q + |Fe_3 - c_\varepsilon|^q \right) + |\gamma_\varepsilon|^p . \tag{8.1}$$

#### Compactness

First let us give some algebraic estimates for $W_\varepsilon(F)$ which reveal the growth behavior with respect to the components $Fe_1$, $Fe_2$ and $Fe_3$.

**Lemma 8.5 (Algebraic estimates)** *Suppose $p$ and $q$ satisfy (P1) and (P2). Let $\varepsilon \in (0,1)$ and $F \in \mathbb{R}^{3\times 3}$. Then,*

*(i)* $\big||Fe_1| - 1\big|^q \leq \varepsilon\, W_\varepsilon(F)$,

*(ii)* $\big||Fe_3| - 1\big|^q \leq \varepsilon\, W_\varepsilon(F)$,

*(iii)* $|Fe_2| \leq \varepsilon^{1/q}\, W_\varepsilon(F)^{\frac{p+q}{pq}} + \varepsilon^{1/q} W_\varepsilon(F)^{1/q} + W_\varepsilon(F)^{1/p} + 1$,

*(iv)* $|(\operatorname{cof} F)e_3 - Fe_3| \leq 2\, \varepsilon^{1/q} \left(W_\varepsilon(F)^{2/q} + 1\right)$,

*(v)* $|\det F - 1| \le 9\,\varepsilon^{1/q}\big(W_\varepsilon(F)^{3/q} + 1\big)$.

*PROOF.* Regarding (8.1) the estimates (i)-(iii) follow as in Lemma 7.2. Next it will be shown that (iv) holds true. To this end we introduce $X_\varepsilon = X_\varepsilon(F) := F(\mathbb{I} - \gamma_\varepsilon e_1 \otimes e_2)$. It holds $X_\varepsilon e_3 = Fe_3$, $X_\varepsilon e_1 = Fe_1$, $(\operatorname{cof} X_\varepsilon)e_3 = X_\varepsilon e_1 \wedge X_\varepsilon e_2 = Fe_1 \wedge (Fe_2 - \gamma_\varepsilon Fe_1) = Fe_1 \wedge Fe_2 = (\operatorname{cof} F)e_3$. Further let $Q_{X_\varepsilon} = \operatorname{argmin}_{Q \in SO(3)} |X_\varepsilon - Q|$. This notation in mind one infers

$$
\begin{aligned}
|(\operatorname{cof} F)e_3 - Fe_3| &= |(\operatorname{cof} X_\varepsilon)e_3 - X_\varepsilon e_3 - (\operatorname{cof} Q_{X_\varepsilon})e_3 + Q_{X_\varepsilon} e_3| \\
&\le |X_\varepsilon e_3 - Q_{X_\varepsilon} e_3| + |X_\varepsilon e_1|\,|X_\varepsilon e_2 - Q_{X_\varepsilon} e_2| + |Q_{X_\varepsilon} e_2|\,|X_\varepsilon e_1 - Q_{X_\varepsilon} e_1| \\
&\le (\sqrt{2} + |Fe_1|)\,|X_\varepsilon - Q_{X_\varepsilon}| \le \Big((\varepsilon W_\varepsilon(F))^{1/q} + 1 + \sqrt{2}\Big)\ \operatorname{dist}(X_\varepsilon, SO(3)) \\
&\le \varepsilon^{1/q}\, W_\varepsilon(F)^{1/q}\big(W_\varepsilon(F)^{1/q} + 1 + \sqrt{2}\big) \ \le\ 2\,\varepsilon^{1/q}\big(W_\varepsilon(F)^{2/q} + 1\big).
\end{aligned}
$$

In order to verify (v), we compute

$$
\begin{aligned}
\det F \;&=\; (Fe_1 \wedge Fe_2)\cdot Fe_3 = \big(Fe_1 \wedge (Fe_2 - \gamma_\varepsilon Fe_1)\big)\cdot Fe_3 \\
&=\; \big((Fe_1 - a_\varepsilon) \wedge (Fe_2 - \gamma_\varepsilon Fe_1) + a_\varepsilon \wedge (Fe_2 - \gamma_\varepsilon Fe_1)\big)\cdot Fe_3 \\
&=\; \big((Fe_1 - a_\varepsilon) \wedge (Fe_2 - \gamma_\varepsilon Fe_1 - b_\varepsilon)\big)\cdot (Fe_3 - (a_\varepsilon \wedge b_\varepsilon)) \\
&\quad + \big((Fe_1 - a_\varepsilon) \wedge (Fe_2 - \gamma_\varepsilon Fe_1 - b_\varepsilon)\big)\cdot (a_\varepsilon \wedge b_\varepsilon) \\
&\quad + ((Fe_1 - a_\varepsilon) \wedge b_\varepsilon)\cdot (Fe_3 - (a_\varepsilon \wedge b_\varepsilon)) + ((Fe_1 - a_\varepsilon) \wedge b_\varepsilon)\cdot (a_\varepsilon \wedge b_\varepsilon) \\
&\quad + (a_\varepsilon \wedge (Fe_2 - \gamma_\varepsilon Fe_1 - b_\varepsilon))\cdot (Fe_3 - (a_\varepsilon \wedge b_\varepsilon)) \\
&\quad + (a_\varepsilon \wedge (Fe_2 - \gamma_\varepsilon Fe_1 - b_\varepsilon))\cdot (a_\varepsilon \wedge b_\varepsilon) + (a_\varepsilon \wedge b_\varepsilon)\cdot (Fe_3 - (a_\varepsilon \wedge b_\varepsilon)) + 1\,.
\end{aligned}
$$

With the help of (8.1) and Young's inequality this implies for $\varepsilon \in (0,1)$ that

$$
|\det F - 1| \le \varepsilon^{1/q}\Big(W_\varepsilon(F)^{3/q} + 3W_\varepsilon(F)^{2/q} + 3W_\varepsilon(F)^{1/q}\Big) \le 9\,\varepsilon^{1/q}(W_\varepsilon(F)^{3/q} + 1).
$$

The proof of the lemma is complete. □

In the next step the special structure of bounded energy sequences $\{u_\varepsilon\}_{\varepsilon>0}$ of $E_\varepsilon$ will be analyzed. Similarly to Section 7.1.3 we define for $\varepsilon > 0$

$$
\begin{aligned}
h_\varepsilon &= (\nabla u_\varepsilon)e_1 - a_\varepsilon\,, \\
k_\varepsilon &= (\nabla u_\varepsilon)e_3 - c_\varepsilon\,, \\
f_\varepsilon &= \gamma_\varepsilon h_\varepsilon = \gamma_\varepsilon\big((\nabla u_\varepsilon)e_1 - a_\varepsilon\big)\,, \\
g_\varepsilon &= (\nabla u_\varepsilon)e_2 - f_\varepsilon = (\nabla u_\varepsilon)e_2 - \gamma_\varepsilon(\nabla u_\varepsilon)e_1 + \gamma_\varepsilon a_\varepsilon = g_\varepsilon^{(1)} + g_\varepsilon^{(2)}, \\
g_\varepsilon^{(1)} &= (\nabla u_\varepsilon)e_2 - \gamma_\varepsilon(\nabla u_\varepsilon)e_1 - b_\varepsilon, \\
g_\varepsilon^{(2)} &= \gamma_\varepsilon a_\varepsilon + b_\varepsilon,
\end{aligned}
$$

where $a_\varepsilon$, $c_\varepsilon$ and $\gamma_\varepsilon$ are given by (8.1). Then there is the additive decomposition of the deformation gradients

$$\nabla u_\varepsilon = A_\varepsilon + H_\varepsilon, \tag{8.2}$$

with $A_\varepsilon = a_\varepsilon \otimes e_1 + g_\varepsilon \otimes e_2 + c_\varepsilon \otimes e_3$ and $H_\varepsilon = h_\varepsilon \otimes e_1 + f_\varepsilon \otimes e_2 + k_\varepsilon \otimes e_3$.

For completeness sake we state here the three-dimensional analogues of Lemma 7.29 and Corollary 7.30.

**Lemma 8.6** *Let $p, q$ satisfy (P1) and (P2) and let $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,1}(\Omega;\mathbb{R}^2)$ be a bounded energy sequence of $E_\varepsilon$, that is $E_\varepsilon[u_\varepsilon] < B$ for all $\varepsilon > 0$. Then there are the estimates*

*(i)* $\|a_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^3)} \leq 1$, *(ii)* $\|c_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^3)} \leq 1$,

*(iii)* $\|h_\varepsilon\|_{L^q(\Omega;\mathbb{R}^3)} \leq (\varepsilon B)^{1/q}$, *(iv)* $\|k_\varepsilon\|_{L^q(\Omega;\mathbb{R}^3)} \leq (\varepsilon B)^{1/q}$,

*(v)* $\|\gamma_\varepsilon\|_{L^p(\Omega)} \leq B^{1/p}$, *(vi)* $\|f_\varepsilon\|_{L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^3)} \leq \varepsilon^{1/q} B^{\frac{p+q}{pq}}$,

*(vii)* $\|g_\varepsilon^{(1)}\|_{L^q(\Omega;\mathbb{R}^3)} \leq (\varepsilon B)^{1/q}$, *(viii)* $\|g_\varepsilon^{(2)}\|_{L^p(\Omega;\mathbb{R}^3)} \leq 2^{\frac{p-2}{2p}}(B + |\Omega|)^{1/p}$.

*Further, $\{A_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^{\min\{p,q\}}(\Omega;\mathbb{R}^{3\times3})$ and $\{H_\varepsilon\}_{\varepsilon>0}$ is equi-integrable and bounded in $L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^{3\times3})$.*

**Corollary 8.7** *Under the assumptions of Lemma 8.6 there exist a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ as $k \to \infty$ and functions $a, c \in L^\infty(\Omega;\mathbb{R}^3)$ and $g \in L^p(\Omega;\mathbb{R}^3)$ with*

*(i)* $a_{\varepsilon_k} \overset{*}{\rightharpoonup} a$ *in* $L^\infty(\Omega;\mathbb{R}^3)$, *(ii)* $c_{\varepsilon_k} \overset{*}{\rightharpoonup} c$ *in* $L^\infty(\Omega;\mathbb{R}^3)$,

*(iii)* $h_{\varepsilon_k} \to 0$ *in* $L^q(\Omega;\mathbb{R}^3)$, *(iv)* $k_{\varepsilon_k} \to 0$ *in* $L^q(\Omega;\mathbb{R}^3)$,

*(v)* $f_{\varepsilon_k} \to 0$ *in* $L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^3)$, *(vi)* $g_{\varepsilon_k}^{(1)} \to 0$ *in* $L^q(\Omega;\mathbb{R}^3)$,

*(vii)* $g_{\varepsilon_k}^{(2)} \rightharpoonup g$ *in* $L^p(\Omega;\mathbb{R}^3)$

*as $k \to \infty$. Moreover, $H_{\varepsilon_k} \to 0$ in $L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^{3\times3})$ and $A_{\varepsilon_k} \rightharpoonup A$ in $L^{\min\{p,q\}}(\Omega;\mathbb{R}^{3\times3})$ with $A = a \otimes e_1 + g \otimes e_2 + c \otimes e_3 \in L^p(\Omega;\mathbb{R}^{3\times3})$.*

*PROOF of Theorem 8.3 (Compactness).* Like in the proof of compactness for Theorem 7.18 and 7.28 we conclude from the algebraic estimates of Lemma 8.5 that a finite energy sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of $E_\varepsilon$ has a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ which converges strongly in $L^{\frac{pq}{p+q}}$ to a function $u \in W^{1,\frac{pq}{p+q}}(\Omega;\mathbb{R}^3)$. Moreover, by calculating weak derivatives similarly to (7.43) one infers that $\nabla u \in W^{1,p}(\Omega;\mathbb{R}^3)$ and that $A = \nabla u$, where $A$ was defined by Corollary 8.7. For proving $\nabla u \in \mathcal{N}^{(3)}$, the estimate $|(\nabla u)e_1| = |a| \leq 1$ almost everywhere in $\Omega$ is an immediate consequence of Corollary 8.7 (i). With respect to the determinant constraint one has the generalized div-curl lemma of Theorem 6.20 as a crucial tool. In particular, we apply it in the form of Corollary 6.26, which was exactly formulated for this purpose. Let us start by writing

$$\begin{aligned} \det \nabla u_{\varepsilon_k} &= (\operatorname{cof} \nabla u_{\varepsilon_k}) e_3 \cdot (\nabla u_{\varepsilon_k}) e_3 \\ &= \big(a_{\varepsilon_k} \wedge g_{\varepsilon_k} + a_{\varepsilon_k} \wedge f_{\varepsilon_k} + h_{\varepsilon_k} \wedge g_{\varepsilon_k} + h_{\varepsilon_k} \wedge f_{\varepsilon_k}\big) \cdot (c_{\varepsilon_k} + k_{\varepsilon_k}) \\ &= \det A_{\varepsilon_k} + (a_{\varepsilon_k} \wedge f_{\varepsilon_k}) \cdot c_{\varepsilon_k} + (a_{\varepsilon_k} \wedge g_{\varepsilon_k}) \cdot k_{\varepsilon_k} + (h_{\varepsilon_k} \wedge g_{\varepsilon_k}) \cdot c_{\varepsilon_k} \\ &\qquad + (h_{\varepsilon_k} \wedge g_{\varepsilon_k}) \cdot k_{\varepsilon_k} + (a_{\varepsilon_k} \wedge f_{\varepsilon_k}) \cdot k_{\varepsilon_k} + (h_{\varepsilon_k} \wedge f_{\varepsilon_k}) \cdot (c_{\varepsilon_k} + k_{\varepsilon_k}), \end{aligned} \tag{8.3}$$

for all $k \in \mathbb{N}$, where the last term vanishes in view of the parallelism of $h_{\varepsilon_k}$ and $f_{\varepsilon_k}$. The second to the forth term converge to zero strongly in $L^{\min\{\frac{pq}{p+q}, \frac{q}{2}\}}$ by Corollary 8.7, because

$$L^{\min\{p,q\}}(\Omega) \cdot L^q(\Omega) \subset L^{\min\{\frac{pq}{p+q}, \frac{q}{2}\}}(\Omega).$$

Due to

$$L^q(\Omega) \cdot L^{\min\{p,q\}}(\Omega) \cdot L^q(\Omega) \subset L^{\min\{\frac{pq}{2p+q}, \frac{q}{3}\}}(\Omega) \quad \text{and} \quad L^{\frac{pq}{p+q}}(\Omega) \cdot L^q(\Omega) \subset L^{\frac{pq}{2p+q}}(\Omega)$$

Corollary 8.7 yields strong $L^{\min\{\frac{pq}{2p+q}, \frac{q}{3}\}}$-convergence of the fifth and sixth term on the right-hand side of (8.3) to zero. Notice that (P3) is equivalent to $\frac{pq}{2p+q} \geq 1$ and that $\frac{q}{3} \geq 1$ by (P2), so that we conclude from (8.3) that

$$\det \nabla u_{\varepsilon_k} - \det A_{\varepsilon_k} \to 0 \quad \text{in} \quad L^1(\Omega) \tag{8.4}$$

as $k \to \infty$. Then, accounting for Corollary 8.7, the equality $A = \nabla u$ and (8.4) the application of Corollary 6.26 implies

$$\det \nabla u_{\varepsilon_k} \rightharpoonup \det \nabla u \quad \text{in} \quad L^1(\Omega). \tag{8.5}$$

Besides on infers from Lemma 8.5 (v),

$$\det \nabla u_{\varepsilon_k} \to 1 \quad \text{in} \quad L^1(\Omega) \quad \text{for} \quad k \to \infty. \tag{8.6}$$

Observe that (P2) is necessary to guarantee $\det \nabla u_{\varepsilon_k} \in L^1(\Omega)$ for $k \in \mathbb{N}$. Hence, by the uniqueness of the limit we obtain as a consequence of (8.5) and (8.6) that $\det \nabla u = 1$ almost everywhere in $\Omega$. This finishes the proof. □

### Upper bound

In this three-dimensional context one cannot simply copy the proof of the 2D situation and start by the seemingly self-evident analogue of Lemma 7.21 or Lemma 7.31. This is because the convex integration result Lemma 5.30, which is the essential ingredient for the latter, is not correct in 3D, compare Remark 7.25. To avoid this problem we make an elementary and direct construction via simple laminates in the interior of a ball using a cut-off function to realize the desired boundary conditions. One has to be careful, though, to keep control on the gradients in the cut-off regions, which in turn have to be made small enough in order to allow for the construction to be continued globally.

**Lemma 8.8** *Let $p$ satisfy (P1) and let $r > 0$, $\mu \in (0,1)$ and $\xi > 0$. Further suppose $u_0(y) = Fy$ for $y \in B(0,r)$ with $F \in \mathcal{N}^{(3)}$. Then there exist functions $\gamma \in L^\infty(B(0,r))$ and $z \in W^{1,\infty}(B(0,r);\mathbb{R}^3)$ such that*

*(i)* $z = u_0$ *on* $\partial B(0,r)$,

*(ii)* $\nabla z(\mathbb{I} - \gamma e_1 \otimes e_2) \in \mathrm{SO}(3)$ *almost everywhere in* $B(0,\mu r)$,

*(iii)* $|\nabla z - F| \leq 3$ *almost everywhere in* $B(0,r)$,

*(iv)* $|\gamma|^p \leq W_p^{\mathrm{rc}}(F)$ *almost everywhere in* $B(0,r)$,

*(v)* $\|z - u_0\|_{L^\infty(B(0,r);\mathbb{R}^3)} \leq \xi$.

*PROOF.* Without loss of generality we may assume $B(0,r) = B(0,1)$. If $F \in \mathcal{M}^{(3)}$, then $z = u_0$ and $\gamma$ given through $\gamma_F$ in $B(0,1)$ have the required properties. Here $\gamma_F$ is defined by the unique representation $F = Q_F(\mathbb{I} + \gamma_F e_1 \otimes e_2)$ with $Q_F \in \mathrm{SO}(3)$. Indeed, $|\gamma|^p = (\gamma_F^2)^{p/2} = (|Fe_2|^2 - 1)^{p/2} = W_p^{\mathrm{rc}}(F)$ in $B(0,1)$.
In the following let $F \in \mathcal{N}^{(3)} \backslash \mathcal{M}^{(3)}$. According to the construction in the proof of Lemma 7.21, which immediately carries over to three dimensions, there are $F^+, F^- \in \mathcal{M}^{(3)}$ and $\lambda \in (0,1)$ such that $F^+ - F^- = a \otimes e_1$ with $a \in \mathbb{R}^3$, $F = \lambda F^+ + (1-\lambda)F^-$, $F^+e_1 \neq F^-e_1$ and $F^+e_2 = F^-e_2$. Now let us define a laminate $l$ between $F^+$ and $F^-$ with weight $\lambda$ and period $h > 0$ (to be chosen later). That is

$$l(y) = u_0(y) + h\,\chi_\lambda\left(\frac{e_1 \cdot y}{h}\right) a,$$

where $\chi_\lambda$ is a continuous, bounded, one-periodic real-valued function of one variable with mean value zero on $(0,1)$ such that

$$\chi_\lambda'(t) = \begin{cases} 1-\lambda & \text{for } t \in (0,\lambda), \\ -\lambda & \text{for } t \in (\lambda,1). \end{cases}$$

It holds that

$$\max_{y \in B(0,1)} |l(y) - u_0(y)| \leq |a|h. \tag{8.7}$$

Next we take $\eta : B(0,1) \to [0,1]$ to be a cut-off function with $\eta \in C_0^\infty(B(0,1))$, $\eta \equiv 1$ on $B(0,\mu)$ and $|\nabla\eta| \le \frac{2}{1-\mu}$. Let us define $z \in W^{1,\infty}(B(0,1);\mathbb{R}^3)$ by

$$z = \eta\,(l - u_0) + u_0$$

and $\gamma \in L^\infty(B(0,1))$ through the unique decomposition $\nabla z(y) = Q(y)(\mathbb{I} + \gamma(y) e_1 \otimes e_2)$ with $Q(y) \in \mathrm{SO}(3)$, if $y \in B(0,\mu)$ and through 0 else. Hence, $z$ and $\gamma$ fulfill (i) and (ii) by construction. In $B(0,\mu)$ we have $|\gamma|^p = (|(\nabla z)e_2|^2 - 1)^{p/2} = (|(\nabla l)e_2|^2 - 1)^{p/2} = (|Fe_2|^2 - 1)^{p/2} = W_p^{\mathrm{rc}}(F)$. In view of the non-negativity of $W_p^{\mathrm{rc}}$ this yields (iv).
In order to prove (iii) let us estimate the term $|\nabla z - F|$ as follows. Note that according to Rademacher's theorem $\nabla z$ can be understood in the sense of classical derivatives almost everywhere in $B(0,1)$. Thus,

$$\begin{aligned} |\nabla z - F| &= \big|(\nabla\eta)(l-u_0) + \eta\,\nabla(l-u_0)\big| \le |\nabla\eta||l-u_0| + |\nabla(l-u_0)e_1| \\ &\le \frac{2}{1-\mu}|l-u_0| + 1 + |Fe_1| \le \frac{2}{1-\mu}|a|h + 2 \end{aligned} \tag{8.8}$$

almost everywhere in $B(0,1)$.
Now $h > 0$ is chosen as $h = \min\left\{\frac{\xi}{|a|}, \frac{1-\mu}{2|a|}\right\}$. Then in view of (8.8) we obtain $|\nabla z - F| \le 3$ almost everywhere in $B(0,1)$ and by (8.7) one finds

$$\|z - u_0\|_{L^\infty(B(0,1);\mathbb{R}^3)} \le \|l - u_0\|_{L^\infty(B(0,1);\mathbb{R}^3)} \le \xi.$$

These are exactly (iii) and (v). □

With this explicit construction at hand we can now establish the elementary building blocks of the recovery sequence. Notice that we choose $\delta$ a little different than in Lemma 7.32. This is just to present another, slightly more elegant version of the proof.

**Lemma 8.9** *Suppose $p, q$ fulfill (P1) and (P4). Let $x \in \mathbb{R}^3$, $\rho > 0$, $u \in W^{1,p}(B(x,\rho);\mathbb{R}^3)$ and $F \in \mathcal{N}^{(3)}$. Moreover, assume that*

$$\delta = \frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} |\nabla u - F|^p + |W_p^{\mathrm{rc}}(\nabla u) - W_p^{\mathrm{rc}}(F)|\,\mathrm{d}y \tag{8.9}$$

*satisfies $\delta < 1$.*
*Then there are $r \in (\rho/2, 3\rho/4)$, $\gamma \in L^p(B(x,r))$ and $w \in W^{1,p}(B(x,r);\mathbb{R}^3)$ such that*

*(i)* $w = u$ *on* $\partial B(x,r)$,

*(ii)* $\displaystyle\int_{B(x,r)} |\gamma|^p\,\mathrm{d}y \le \int_{B(x,r)} W_p^{\mathrm{rc}}(\nabla u)\,\mathrm{d}y + 8|B(x,r)|\,\delta,$

*(iii)* $\displaystyle\int_{B(x,r)} \mathrm{dist}^q\big(\nabla w(\mathbb{I} - \gamma e_1 \otimes e_2), \mathrm{SO}(3)\big)\,\mathrm{d}y \le c(p,q)|B(x,r)|\big(W_p^{\mathrm{rc}}(F)^{q/p} + 1\big)\,\delta^{q/p},$

*(iv)* $\|w-u\|_{L^{\frac{pq}{p+q}}(B(x,r);\mathbb{R}^3)} \leq c(p,q)\,r\,|B(x,r)|^{\frac{p+q}{pq}}\,\delta^{1/p}$.

*PROOF.* After a simple scaling argument it is sufficient to consider the problem on $B(0,1)$ and assume $u_{B(0,1)}=0$. Let $u_0(y)=Fy$ for $y\in B(0,1)$. Besides, from now on we focus on $q>p$. The case $q\leq p$ follows similarly, but is much easier, because all the effort put into gaining higher integrability is not needed there.
Now we recall Lemma 6.1 and choose $r\in(1/2,3/4)$ in such a way that $u-u_0\in W^{1,p}(\partial B(0,r);\mathbb{R}^3)$ together with the estimate

$$\|u-u_0\|^p_{W^{1,p}(\partial B(0,r);\mathbb{R}^3)} \leq c(p)\;\|\nabla u-F\|^p_{L^p(B(0,1);\mathbb{R}^{3\times 3})} \leq c(p)\delta. \tag{8.10}$$

By $\gamma\in L^\infty(B(0,r))$ and $z\in W^{1,\infty}(B(0,r);\mathbb{R}^3)$ we denote the functions constructed on $B(0,r)$ with the help of Lemma 8.8 taking $\xi=\delta^{1/p}$ and $\mu\in(0,1)$, which will be chosen later. Further, let $v$ stand for the harmonic extension of $u$ restricted to $\partial B(0,r)$ on $B(0,r)$, i.e. $v$ is the solution of $\Delta v=0$ in $B(0,r)$ with $v=u$ on $\partial B(0,r)$. We define

$$w(y)=v(y)-u_0(y)+z(y) \qquad \text{for } y\in B(0,r).$$

Regarding the proof of (ii) we calculate under consideration of Lemma 8.8 (iv),

$$\begin{aligned}\int_{B(0,r)}|\gamma|^p\,\mathrm{d}y &\leq \int_{B(0,r)} W_p^{\mathrm{rc}}(F)\,\mathrm{d}y\\ &\leq \int_{B(0,r)} W_p^{\mathrm{rc}}(\nabla u)\,\mathrm{d}y+\int_{B(0,r)}\left|W_p^{\mathrm{rc}}(F)-W_p^{\mathrm{rc}}(\nabla u)\right|\mathrm{d}y\\ &\leq \int_{B(0,r)} W_p^{\mathrm{rc}}(\nabla u)\,\mathrm{d}y+8|B(0,r)|\,\delta.\end{aligned}$$

Next we want to show (iii). Let $Q$ be given by $Q=\nabla z(\mathbb{I}-\gamma e_1\otimes e_2)\in\mathrm{SO}(3)$ in $B(0,\mu r)$ and observe

$$\begin{aligned}\operatorname{dist}\big(\nabla w(\mathbb{I}-\gamma e_1\otimes e_2),\mathrm{SO}(3)\big) &\leq \big|\nabla w(\mathbb{I}-\gamma e_1\otimes e_2)-Q\big|\\ &\leq |\mathbb{I}-\gamma e_1\otimes e_2|\,|\nabla w-\nabla z| = |\mathbb{I}-\gamma e_1\otimes e_2|\,|\nabla v-F|.\end{aligned}$$

With $|\mathbb{I}-\gamma e_1\otimes e_2|^2=\gamma^2+3\leq W_p^{\mathrm{rc}}(F)^{2/p}+3$, which follows from Lemma 8.8 (iv), we infer

$$|\mathbb{I}-\gamma e_1\otimes e_2|^q \leq \big(W_p^{\mathrm{rc}}(F)^{2/p}+3\big)^{q/2} \leq c(q)\big(W_p^{\mathrm{rc}}(F)^{q/p}+1\big). \tag{8.11}$$

Moreover, using Lemma 6.3, (8.10) and Hölder's inequality one has

$$\begin{aligned}\|\nabla v-F\|^q_{L^q(B(0,r);\mathbb{R}^{3\times 3})} &\leq c(p,q)\|v-u_0\|^q_{W^{1,3p/2}(B(0,r);\mathbb{R}^3)}\\ &\leq\; c(p,q)\,\|u-u_0\|^q_{W^{1,p}(\partial B(0,r);\mathbb{R}^3)} \leq\; c(p,q)\,|B(0,r)|\,\delta^{q/p},\end{aligned} \tag{8.12}$$

since $q \leq \frac{3}{2}p$ by (P4) and $r \in (1/2, 3/4)$. For the remaining annulus $B(0,r) \setminus B(0,\mu r)$ we estimate in view of $\gamma = 0$ that

$$\begin{aligned}
\int_{B(0,r)\setminus B(0,\mu r)} \operatorname{dist}^q\big(\nabla w(\mathbb{I} - \gamma e_1 \otimes e_2), \mathrm{SO}(3)\big)\, \mathrm{d}y &\leq c(q) \int_{B(0,r)\setminus B(0,\mu r)} |\nabla w|^q + 1\, \mathrm{d}y \\
&\leq c(q) \int_{B(0,r)\setminus B(0,\mu r)} |\nabla v - F|^q + |\nabla z - F|^q + \big(|F|^q + 1\big)\, \mathrm{d}y \\
&\leq c(p,q)\Big(|B(0,r)|\delta^{q/p} + |B(0,r) \setminus B(0,\mu r)|(|F|^q + 1 + 3^q)\Big) \qquad (8.13) \\
&\leq c(p,q)|B(0,r)|\Big(\delta^{q/p} + (|F|^q + 1)(1 - \mu^3)\Big),
\end{aligned}$$

where we essentially used Theorem 8.8 (iii) and (8.12). After choosing $\mu$ close enough to one (iii) is a consequence of (8.11), (8.12) and (8.13). It remains to verify (v). Here we exploit Lemma 8.8 (v), Poincaré's and Hölder's inequality and (8.12) to get

$$\begin{aligned}
\|w - u\|_{L^{\frac{pq}{p+q}}(B(0,r);\mathbb{R}^3)} &\leq \|z - u_0\|_{L^{\frac{pq}{p+q}}(B(0,r);\mathbb{R}^3)} \\
&\qquad + \|u - u_0\|_{L^{\frac{pq}{p+q}}(B(0,r);\mathbb{R}^3)} + \|v - u_0\|_{L^{\frac{pq}{p+q}}(B(0,r);\mathbb{R}^3)} \\
&\leq |B(0,r)|^{\frac{p+q}{pq}}\, \delta^{1/p} + c(p,q)\, \|\nabla u - F\|_{L^p(B(0,1);\mathbb{R}^{3\times 3})} + c(p,q)\, \|v - u_0\|_{W^{1,3p/2}(B(0,r);\mathbb{R}^3)} \\
&\leq c(p,q)\, r\, |B(0,r)|^{\frac{p+q}{pq}}\, \delta^{1/p}.
\end{aligned}$$

This concludes the proof. $\square$

Here is the next move to constructing an appropriate recovery sequence, namely the three-dimensional parallel of Lemma 7.33. Again, this is the key step, because it makes it possible to perform the basic construction from above on a fixed volume percentage of the set $\Omega \setminus G$ where no local construction has been applied up to then.

**Lemma 8.10** *Let $p$ and $q$ satisfy (P1) and (P4). There exists a $\theta \in (0,1)$ with the following properties: Let $\Omega \subset \mathbb{R}^3$ be a bounded, open set and $G \subset \Omega$ be a closed subset. For every $u \in W^{1,p}(\Omega;\mathbb{R}^3)$ with $\nabla u \in \mathcal{N}^{(3)}$ almost everywhere on $\Omega \setminus G$ and every $\xi > 0$ there are a closed set $\omega \subset \Omega \setminus G$ and functions $w \in W^{1,p}(\Omega;\mathbb{R}^3)$ and $\gamma \in L^p(\omega)$ such that*

*(i)* $w = u$ *on* $\Omega \setminus \omega$,

*(ii)* $|\omega| \geq \theta|\Omega \setminus G|$,

*(iii)* $\displaystyle\int_\omega |\gamma|^p\, \mathrm{d}y \leq \int_\omega W_p^{\mathrm{rc}}(\nabla u)\, \mathrm{d}y + \xi|\omega|$,

*(iv)* $\displaystyle\int_\omega \operatorname{dist}^q\big(\nabla w(\mathbb{I} - \gamma e_1 \otimes e_2), \mathrm{SO}(3)\big)\, \mathrm{d}y \leq \xi|\omega|$,

*(v)* $\|u-w\|^{\frac{pq}{p+q}}_{L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^3)} \le \xi|\omega|$.

*PROOF.* Let $\Omega_L$ be the intersection of the set of Lebesgue points for $\nabla u$ and $W_p^{\mathrm{rc}}(\nabla u)$ in $\Omega \setminus G$ and the set $\{x \in \Omega\setminus G \,|\, \nabla u(x) \in \mathcal{N}^{(3)}\}$. For $\eta \in (0,1)$ (to be chosen later) we define $\delta : \Omega_L \to (0,1)$ by

$$\delta(x) = \frac{\eta}{\left(1 + W_p^{\mathrm{rc}}(\nabla u(x))^{q/p}\right)^{p/q}}\,.$$

Since $\Omega\setminus G$ is open, for every $x \in \Omega_L$ there exists a $\rho(x) \in (0,1)$ with $\overline{B(x,\rho(x))} \subset \Omega\setminus G$ and

$$\int_{B(x,\rho(x))} |\nabla u(y) - \nabla u(x)|^p + \left|W_p^{\mathrm{rc}}(\nabla u(y)) - W_p^{\mathrm{rc}}(\nabla u(x))\right| \,\mathrm{d}y \le |B(x,\rho(x))|\,\delta(x). \quad (8.14)$$

By Vitali's covering theorem there is an at most countable set $A \subset \Omega_L$ such that the balls of the family $\{\overline{B(x,\rho(x))}\}_{x\in A}$ are pairwise disjoint and satisfy

$$\sum_{x\in A} |B(x,\rho(x))| \ge \frac{1}{25}\,|\Omega\setminus G|\,. \quad (8.15)$$

Then one can choose finitely many of these points $\{x_1,\dots,x_N\} \subset A$ such that

$$\sum_{j=1}^{N} |B(x_j,\rho_j)| \ge \frac{1}{30}\,|\Omega\setminus G|\,, \quad (8.16)$$

with the notation $\rho_j = \rho(x_j)$ for $j \in \{1,\dots,N\}$. In the following let $r_j$ denote the radii $r_j \in (\rho_j/2, 3\rho_j/4)$ emerging from Lemma 8.9 applied to $B(x_j,\rho_j)$ with $F = \nabla u(x_j)$. Let $w$ be the function which agrees on each of the balls $B(x_j,r_j)$ with the corresponding function resulting from Lemma 8.9 and equals $u$ outside $\bigcup_{j=1}^N B(x_j,r_j)$. We define $\gamma$ analogously as the function which coincides on each ball $B(x_j,r_j)$ with the function $\gamma_j$ obtained by Lemma 8.9 and set

$$\omega = \bigcup_{j=1}^{N} \overline{B(x_j,r_j)}.$$

With (8.16) and $\theta = 1/240$ it holds

$$|\omega| \;=\; \sum_{j=1}^{N} |B(x_j,r_j)| \ge \frac{1}{2^3}\sum_{j=1}^{N} |B(x_j,\rho_j)| \ge \frac{1}{8}\frac{1}{30}|\Omega\setminus G| = \theta|\Omega\setminus G|,$$

which is assertion (ii). Accounting for (8.14) we may estimate

$$\begin{aligned}\int_\omega \operatorname{dist}^q\big(\nabla w(\mathbb{I}-\gamma e_1\otimes e_2),\mathrm{SO}(3)\big)\,\mathrm{d}y \\ &\le c(p,q)\sum_{j=1}^N |B(x_j,r_j)|\big(W_p^{\mathrm{rc}}(\nabla u(x_j))\big)^{q/p}+1)\delta(x_j)^{q/p} \\ &\le\ c(p,q)\,\eta^{q/p}\sum_{j=1}^N|B(x_j,r_j)| = c(p,q)|\omega|\eta^{q/p}.\end{aligned}$$

The remaining calculations to get (iii) and (v) are analogous to Lemma 7.33 and Lemma 7.23. All the assertions follow finally by choosing $\eta$ small enough. $\square$

From now on the proof is identical to the one of Theorem 7.28. We state the lemma leading to the global result for completeness' sake.

**Lemma 8.11** *Let $\Omega\subset\mathbb{R}^3$ be an open and bounded set and let $p,q$ satisfy (P1) and (P4). For every $u\in W^{1,p}(\Omega;\mathbb{R}^3)$ such that $\nabla u\in\mathcal{N}^{(3)}$ almost everywhere and every $\xi>0$ there are functions $w\in W^{1,\frac{pq}{p+q}}(\Omega;\mathbb{R}^3)$ and $\gamma\in L^p(\Omega)$ such that*

*(i)* $\|u-w\|^{\frac{pq}{p+q}}_{L^{\frac{pq}{p+q}}(\Omega;\mathbb{R}^3)}\le\xi\,|\Omega|,$

*(ii)* $\displaystyle\int_\Omega|\gamma|^p\,\mathrm{d}y\le\int_\Omega W_p^{\mathrm{rc}}(\nabla u)\,\mathrm{d}y+\xi\,|\Omega|,$

*(iii)* $\displaystyle\int_\Omega\operatorname{dist}^q\big(\nabla w(\mathbb{I}-\gamma e_1\otimes e_2),\mathrm{SO}(3)\big)\,\mathrm{d}y\le\xi\,|\Omega|.$

*PROOF of Theorem 8.3 (Upper bound inequality).* All in all one can infer the claim of Theorem 8.3 from the previous lemma. To this end we consider $\{w_\varepsilon\}_{\varepsilon>0}$ which are obtained by setting $\xi=\xi(\varepsilon)=\varepsilon^2$ and follow the lines of the proof of Theorem 7.18. $\square$

# 9. Outlook

There is a number of interesting and promising directions to continue research on this topic and related issues. Let us conclude this thesis by stating some of the open questions in this field. The overview we are giving below is by no means meant to be exhaustive, but rather provides a list of possible future projects, most of which are planned to be addressed by the DFG-research unit 797 'Analysis and computation of microstructure in finite plasticity'.

**Regularization.** Physical experiments on materials forming microstructure usually show oscillations on a defined length scale. Indeed, smaller and smaller patterns are energetically less favorable so that one specific fine structure is eventually attained. From a general viewpoint this is due to the finite quality of nature. In crystal plasticity energetic penalization of microstructure arises from line energies of geometrically necessary dislocations. Mathematically, these can be accounted for by adding a regularizing expression to the present energy density. It consists of a higher order term involving first derivatives of $F$ in form of the geometric dislocation tensor $G = \mathcal{G}(F_{\mathrm{pl}})$. More precisely, for $\delta > 0$ one has the energy contribution

$$\delta \int_\Omega |\operatorname{curl}_2 F_{\mathrm{pl}}| \,\mathrm{d}x \quad \text{and} \quad \delta \int_\Omega \big|(\operatorname{curl}_3 F_{\mathrm{pl}}) F_{\mathrm{pl}}^T\big| \,\mathrm{d}x$$

in two and three space dimensions, respectively. For further modeling aspects regarding dislocations see [56, 37, 38, 23]. This procedure finally renders the problem at hand nonlocal so that it gains an intrinsic length scale, the size of which is controlled by the parameter $\delta$.

In order to approach the problem of regularization one might start by considering the regularized variational problem associated with the elastically rigid single-slip model discussed in Section 4.1. That is, one needs to study

$$\min \big\{ E_\delta[u] \mid u(x) = Fx \text{ on } \partial\Omega \big\}, \tag{9.1}$$

where

$$E_\delta[u] = \int_\Omega W(\nabla u) + \delta |\operatorname{curl}_2 F_{\mathrm{pl}}| \,\mathrm{d}x$$

and $F \in \mathbb{R}^{2\times 2}$ imposes affine boundary conditions. Here we only mention the 2D situation, but the analogous problem can be formulated in three dimensions, as well.

First, there is the still unsolved question of existence of minimizers for (9.1). Since the new term features only linear growth, a rigorous proof is quite challenging, for $\operatorname{curl} F_{\text{pl}}$ need not exist in the classical sense but is rather a measure. So it might be helpful for a start to introduce higher exponents and to replace the curl-term by $|\nabla F_{\text{pl}}|$ in order to get a first idea of how to proceed. A second issue worth analyzing is the optimal scaling behavior of $E_\delta$ with respect to $\delta$. Notice that the infimum of the functional $E_\delta$ converges to $|\Omega| W^{\text{qc}}(F)$ as $\delta$ tends to zero. The rate of convergence, however, depends essentially on the microstructure patterns under consideration. Hence, investigating the scaling of the optimal energy can help to distinguish different patterns and to detect new material structures apart from laminates. A final even though demanding goal could be to find the $\Gamma$-limit of $E_\delta$ as $\delta \to 0$ after suitable scaling.

**Time evolution and relaxation.** If one wants to understand time evolution of elasto-plastic bodies within the time-discrete variational formulation, one actually has to treat the whole sequence of incremental problems and not only the one of the first time step, as it is done throughout this work. The energy density in every time step depends highly on the deformation history and hence on the solution of the previous minimization problem. This may render the condensed energy densities and consequently relaxation more complicated.

Let us briefly go back to the simplified model with rigid elasticity, for which the relaxation is known explicitly. In this context Conti and Theil [24] developed a relaxation scheme and applied it to the special situation of a simple shear test. This way they achieved an exact relaxation for all time steps and were able to calculate approximate solutions to the evolution problem. However, their argumentation does not work for general problems. A future task is to improve this rather restrictive scheme by understanding the connection between the geometry of microstructure and evolution processes.

Recently Mielke and Ortiz [57] came up with a time-continuous reformulation of evolution processes. They introduced a functional whose minimizers characterize the complete trajectories of the system and weighted it with Paretto weights of the form $e^{-t/\varepsilon}$ to regain causality in the limit $\varepsilon \to 0$. Regarding relaxation these problems will require essentially new methods going beyond quasiconvexity. In contrast to the incremental approach, where the problems to deal with resemble static ones in every single time step, this new situation is qualitatively different.

**Polycrystals and homogenization.** So far our investigations have been limited to single crystals in order to avoid further difficulties. Nevertheless, most materials used in applications display a polycrystalline structure, meaning that they have an additional level of micro-patterns in between the macroscale and the fine scale, called the grains. Since the grain boundaries impose restrictions on the still finer structures, the large-scale behavior of a sample is strongly affected by the grains. Homogenization, see [52, 17, 12], could be a helpful tool to understand these effects. If we assume that the polycrystal is composed of periodic cells, the approximate macroscopic behavior can be obtained by solving a family of cell problems. A possible first step in applying homogenization to single-slip models is to focus on the simple case with only two grains separated by one interface. Then, joining several of these elementary components together successively seems a promising approach to analyze bicrystals.

**Models with two active slip systems.** The question of whether the results presented in this thesis extend to models with two or more active slip systems is still open. One of the fundamental problems which prevents transferring the previous arguments is that the relaxations of the corresponding elastically rigid cases are up to now unknown. In [3] the polyconvex and rank-one convex envelopes were determined in the setting of two slip systems. But it turned out that the latter are not identical and hence do not yield the desired relaxation formula. So the essential first task would be to prove the conjecture that the macroscopic energy density coincides with the rank-one convexification.

**Problems with $(p,q)$-growth.** In a wider context the variational problems presented in this thesis can be seen as a special case of problems with $(p,q)$-growth, which we call here problems of $(r,t)$-growth in order to avoid confusion with the elastic and plastic exponents of $W_\varepsilon$, see Remark 7.3. Under the assumption of linear hardening and quadratic elastic energy the following estimates providing coercivity and growth conditions for $W_\varepsilon$ with $\varepsilon > 0$ were shown in Section 7.1.1,

$$c_1|F| - c_2 \leq W_\varepsilon(F) \leq C\,(1 + |F|^2)$$

for all $F \in \mathbb{R}^{2\times 2}$ with constants $c_1, c_2, C > 0$. Here the gap between the order of upper and lower bound is too large for general relaxation theory and standard lower semicontinuity results to apply [33, 46]. Nevertheless, there are additional properties of $W_\varepsilon$ to compensate the lack of good growth conditions. From a mathematical viewpoint it would certainly be interesting to find a way to weaken the requirements concerning the distance between $r$ and $t$ for general problems with non-standard growth by imposing alternative postulates on the density.

# A. Notation

This appendix was added to give the reader an overview of the notation used throughout this thesis and to summarize the most important quantities occurring in the previous chapters.
Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with the scalar product $x \cdot y = \sum_{i=1}^n x_i y_i$ for $x, y \in \mathbb{R}^n$ and the norm $|x| = \sqrt{x \cdot x}$. Depending on the context $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is regarded as a row or a column vector. By $\mathbb{R}^{m\times n}$ we denote the space of real $(m \times n)$-matrices endowed with the norm $|F|^2 = F : F$, where $F : G = \sum_{i=1}^m \sum_{j=1}^n F_{ij} G_{ij}$ is the inner product in $\mathbb{R}^{m\times n}$. For the components of vectors and matrices we always write subscript indices.
In case that $m = n$, the cofactor matrix $\operatorname{cof} F \in \mathbb{R}^{n\times n}$ is defined as the matrix of all $(n-1)\times(n-1)$ minors of $F \in \mathbb{R}^{n\times n}$ which fulfills the equality

$$\operatorname{cof} F = (\det F) F^{-T} \quad \text{or equivalently} \quad (\operatorname{cof} F) F^T = (\det F)\mathbb{I} \tag{A.1}$$

with $\mathbb{I} = \operatorname{diag}(1, \ldots, 1)$ the identity matrix in $\mathbb{R}^{n\times n}$ and $F^T$ the transpose of $F$. In particular, $(A.1)$ provides a way to rewrite the determinant of $F$ as

$$\det F = e_i^T (\operatorname{cof} F) \cdot e_i^T F$$

with $i \in \{1, \ldots, n\}$, where $e_1, \ldots, e_n$ is the standard basis in $\mathbb{R}^n$. Expressions of the form $\mathbb{I} + x \otimes y$ with $x, y \in \mathbb{R}^n$ are frequently used in this thesis, so that it is worth to state some of their basic properties. To start with, $x \otimes y$ is the symbol of the outer product in $\mathbb{R}^n$, meaning $(x \otimes y)_{ij} = x_i y_j$ for $i, j \in \{1 \ldots, n\}$. Obviously $x \otimes y$ represents a rank-one matrix in $\mathbb{R}^{n\times n}$ and it holds

$$(\mathbb{I} + x \otimes y)^{-1} = \mathbb{I} - x \otimes y,$$

if $x \cdot y = 0$. Further one can compute

$$\det(\mathbb{I} + x \otimes y) = 1 + x \cdot y.$$

Hence, we obtain $\det(\mathbb{I} + x \otimes y) = 1$ provided $x \perp y$.
Unless stated otherwise $\Omega \subset \mathbb{R}^n$ is an open and bounded set. We use the standard notation for classical Lebesgue and Sobolev spaces and their norms. Apart from that vector-valued functions are defined componentwise. Using the example of Lebesgue spaces this

means that we identify $L^p(\Omega;\mathbb{R}^m) \cong [L^p(\Omega)]^m$. For all the other function spaces we proceed analogously. To avoid confusion we apply the convention that the range is always noted down explicitly for all function spaces unless we are dealing with real-valued functions. So we write $L^p(\Omega)$ instead of $L^p(\Omega;\mathbb{R})$ for simplicity to stick with the example of Legesgue spaces. For complex-valued functions we use the analogous notation and write $L^p(\Omega) = L^p(\Omega;\mathbb{C})$ provided it is clear from the context what is meant.
For a function $u : \Omega \to \mathbb{R}^m,\ x \mapsto u(x)$ we take $\partial_j u$ to be its (weak) partial derivative with respect to $x_j$ and denote its gradient by $\nabla u$, where $(\nabla u)_{ij} = \partial_j u_i$ with $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. Moreover, the differential operators $\Delta$, div and curl are given through $(\Delta u)_i = \Delta u_i = \sum_{j=1}^n \partial_j \partial_j u_i$ for $i \in \{1,\dots,m\}$, and requiring $n = m$ through $\operatorname{div} u = \sum_{i=1}^n \partial_i u_i$ and $(\operatorname{curl} u)_{ij} = \partial_i u_j - \partial_j u_i$ with $i,j \in \{1,\dots,n\}$. Hence, $\operatorname{curl} u$ is an antisymmetric tensor of order two.
Finally, let us make a brief remark concerning constants. We use the letters $C$ and $c$ to denote any constant that can be determined from the known quantities. Therefore the exact values of $C$ and $c$ may be different from line to line.

For the sake of clarity the next few pages contain a detailed thematic classification of all our notation in tabular form.

*Numbers, vectors and matrices*

| | |
|---|---|
| $\overline{\mathbb{R}}$ | extended real numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ |
| $\mathbb{R}_0^+$ | non-negative real numbers, $\mathbb{R}_0^+ = \{a \in \mathbb{R} \,\vert\, a \geq 0\}$ |
| $\operatorname{sign} a$ | signum of $a \in \mathbb{R}$,<br>$\operatorname{sign} a = 1$ if $a > 0$, $\operatorname{sign} a = 0$ if $a = 0$ and $\operatorname{sign} a = -1$ else |
| $\delta_{ij}$ | Kronecker delta, $i, j \in \mathbb{N}$, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ else |
| $\mathbb{R}^n$ | $n$-dimensional Euclidean space, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ |
| $x \cdot y$ | inner product in $\mathbb{R}^n$, $x \cdot y = \sum_{i=1}^n x_i y_i$ |
| $\vert x\vert$ | Euclidean norm in $\mathbb{R}^n$, $\vert x\vert^2 = x \cdot x$ |
| $\mathbb{R}^{m\times n}$ | space of real $m \times n$ matrices, $F = (F_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \in \mathbb{R}^{m\times n}$ |
| id | identical mapping on $\mathbb{R}^{m\times n}$ |
| $F^{-1}$ | inverse matrix of $F \in \mathbb{R}^{n\times n}$ |
| $F^T$ | transposed matrix of $F \in \mathbb{R}^{n\times n}$ |
| $\operatorname{tr} F$ | trace of $F \in \mathbb{R}^{n\times n}$ |
| $F : G$ | inner product in $\mathbb{R}^{m\times n}$, $F : G = \operatorname{tr}(F^T G)$ |
| $\vert F\vert$ | Euclidean norm in $\mathbb{R}^{m\times n}$, $\vert F\vert^2 = F : F$ |
| $\mathbb{I}$ | identity matrix in $\mathbb{R}^{n\times n}$ |
| $\det F$ | determinant of $F \in \mathbb{R}^{n\times n}$ |
| $\mathrm{O}(n)$ | group of orthogonal matrices $F \in \mathbb{R}^{n\times n}$ |
| $\mathrm{SO}(n)$ | group of orthogonal matrices $F \in \mathbb{R}^{n\times n}$ with $\det F = 1$ |
| $\mathrm{Sl}(n)$ | group of matrices $F \in \mathbb{R}^{n\times n}$ with $\det F = 1$ |
| $M(F)$ | vector of all minors of $F \in \mathbb{R}^{m\times n}$ |
| $\tau(m, n)$ | length of $M(F)$ with $F \in \mathbb{R}^{m\times n}$ |
| $\operatorname{cof} F$ | cofactor matrix of $F \in \mathbb{R}^{n\times n}$, $(\operatorname{cof} F)F^T = (\det F)\mathbb{I}$ |
| $e_1, e_2 \ldots, e_n$ | standard basis in $\mathbb{R}^n$ |
| $x \otimes y$ | rank-one matrix resulting from the outer product in $\mathbb{R}^n$,<br>$(x \otimes y)_{ij} = x_i y_j$ for $i, j \in \{1, \ldots, n\}$ |
| $J$ | counterclockwise rotation by $\pi/2$ in the plane, $J = -e_1 \otimes e_2 + e_2 \otimes e_1$ |
| $x^\perp$ | counterclockwise rotation of $x \in \mathbb{R}^2$ by $\pi/2$ in the plane, $x^\perp = Jx$ |
| $x \wedge y$ | wedge product in $\mathbb{R}^3$,<br>$(x \wedge y)_i = x_j y_k - x_k y_j$ for a cyclic permutation $(ijk)$ of $(123)$ |

*Sets and measures*

| | |
|---|---|
| $B(x,r)$ | open ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r > 0$ |
| $\mathbb{S}^n$ | unit sphere in $\mathbb{R}^{n+1}$ |
| $\partial E$ | boundary of an open and bounded set $E \subset \mathbb{R}^n$ |
| $\overline{E}$, $\overline{\Sigma}$ | closure of a set $E \subset \mathbb{R}^n$ or $\Sigma \subset \mathbb{R}^{m\times n}$ |
| $\operatorname{dist}(E,\tilde{E})$ | distance between the sets $E, \tilde{E} \subset \mathbb{R}^n$ |
| $\operatorname{dist}(F,\Sigma)$ | distance between $F \in \mathbb{R}^{m\times n}$ and the set $\Sigma \subset \mathbb{R}^{m\times n}$ |
| $\mathcal{L}^n(E)$ | $n$-dimensional Lebesgue measure of $E \subset \mathbb{R}^n$ |
| $\lvert E\rvert$ | volume of $E \subset \mathbb{R}^n$, $\lvert E\rvert = \mathcal{L}^n(E)$ |
| $\omega_n$ | volume of the unit ball in $\mathbb{R}^n$ |
| $U \subset\subset E$ | $U \subset \mathbb{R}^n$ is compactly contained in $E \subset \mathbb{R}^n$,<br>i.e. $U \subset \overline{U} \subset E$ with $\overline{U}$ compact |
| $\mathcal{H}^k(E)$ | $k$-dimensional Hausdorff-measure of $E \subset \mathbb{R}^n$ |
| $\chi_E$ | characteristic function of a set $E \subset \mathbb{R}^n$,<br>$\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \neq E$ |
| $\chi$ | extended-valued characteristic function defined in Section 2.2 |
| $\mathcal{M}(\mathbb{R}^{m\times n})$ | space of signed, regular measures on $\mathbb{R}^{m\times n}$, $\mathcal{M}(\mathbb{R}^{m\times n}) = \big(C_0^0(\mathbb{R}^{m\times n})\big)'$ |
| $\mathcal{PM}(\mathbb{R}^{m\times n})$ | space of probability measures on $\mathbb{R}^{m\times n}$, $\mathcal{PM}(\mathbb{R}^{m\times n}) \subset \mathcal{M}(\mathbb{R}^{m\times n})$ |
| $\Sigma^{\mathrm{c}}$ | convex hull of $\Sigma \subset \mathbb{R}^{m\times n}$, see Definition 3.14 |
| $\Sigma^{\mathrm{qc}}$ | quasiconvex hull of $\Sigma \subset \mathbb{R}^{m\times n}$, see Definition 3.15 |
| $\Sigma^{\mathrm{pc}}$ | polyconvex hull of $\Sigma \subset \mathbb{R}^{m\times n}$, see Definition 3.15 |
| $\Sigma^{\mathrm{rc}}$ | rank-one convex hull of $\Sigma \subset \mathbb{R}^{m\times n}$, see Definition 3.15 |
| $\Sigma^{\mathrm{lc}}$ | lamination convex hull of $\Sigma \subset \mathbb{R}^{m\times n}$, see Definition 3.16 |

*Functions and function spaces*

| | |
|---|---|
| $V'$ | dual of a normed vector space $V$ |
| $\langle v', v\rangle_{V',V}$ | duality pairing between $v' \in V'$ and $v \in V$ |
| $\langle v', v\rangle$ | duality pairing between $v' \in V'$ and $v \in V$ in case that the applied spaces are clear from the context |
| $(X, d)$ | metric space $X$ with metric $d$ |
| $C^1([a,b])$ | space of continuously differentiable functions on the interval $[a,b] \subset \mathbb{R}$ |
| $C^\infty(\Omega)$ | space of infinitely differentiable functions on $\Omega$ |
| $C_0^\infty(\Omega)$ | space of test functions on $\Omega$, i.e. space of infinitely differentiable functions with compact support |
| $C_0^0(\mathbb{R}^n)$ | space of continuous functions on $\mathbb{R}^n$ converging to zero at infinity |
| $C_0^0(\mathbb{R}^{m\times n})$ | space of continuous functions on $\mathbb{R}^{m\times n}$ converging to zero at infinity |
| $C_{\mathrm{loc}}^{k,\alpha}(\mathbb{R}^{m\times n})$ | space of $k$-times continuously differentiable functions on $\mathbb{R}^{m\times n}$ whose $k$-th partial derivatives are locally Hölder continuous with exponent $\alpha$ |
| $L^p(\Omega)$ | Lebesgue space on $\Omega$, $1 \le p \le \infty$ |
| $L^p_w(\Omega)$ | weak-$L^p$ space on $\Omega$, $1 \le p \le \infty$, see Definition 5.19 |
| $W^{k,p}(\Omega)$ | Sobolev space on $\Omega$, $k \in \mathbb{N}$, $1 \le p \le \infty$ |
| $W_0^{k,p}(\Omega)$ | Sobolev space with zero boundary condition on $\Omega$, $k \in \mathbb{N}$, $1 \le p \le \infty$ |
| $W_v^{k,p}(\Omega)$ | Sobolev space on $\Omega$ with the boundary values of $v \in W^{k,p}(\Omega)$, $k \in \mathbb{N}$, $1 \le p \le \infty$, $W_v^{k,p}(\Omega) := \{u \in W^{k,p}(\Omega) \mid u - v \in W_0^{k,p}(\Omega)\}$ |
| $W_{\mathrm{loc}}^{k,p}(\Omega)$ | local Sobolev space on $\Omega$, $k \in \mathbb{N}$, $1 \le p \le \infty$ |
| $W^{-k,q}(\Omega)$ | dual space of $W_0^{k,p}(\Omega)$, $k \in \mathbb{N}$, $1 < p < \infty$, $1/q + 1/p = 1$ |
| $W^{-1,1}(\Omega)$ | dual space of $W_0^{1,\infty}(\Omega)$ |
| $W_p^s(\Omega)$ | Sobolev-Slobodeckij space on $\Omega$, $s \ge 0$, $1 < p < \infty$, see Definition 5.13 |
| $B_{p,q}^s(\Omega)$ | Besov space on $\Omega$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q < \infty$, see Definition 5.13 |
| $H_p^s(\Omega)$ | Bessel-potential space on $\Omega$, $s \in \mathbb{R}$, $1 < p < \infty$, see Definition 5.13 |
| $\mathcal{D}'(\Omega)$ | space of distributions on $\Omega$, dual of $C_0^\infty(\Omega)$ |
| $\delta_x$ | Dirac measure giving unit mass to the point $x \in \Omega$, $\langle \delta_x, \varphi\rangle_{\mathcal{D}'(\Omega), C_0^\infty(\Omega)} = \varphi(x)$ for all $\varphi \in C_0^\infty(\Omega)$ |
| $\mathcal{S}(\mathbb{R}^n)$ | Schwartz space of complex-valued rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^n$ |
| $\mathcal{S}'(\mathbb{R}^n)$ | space of tempered distributions on $\mathbb{R}^n$, dual of $\mathcal{S}(\mathbb{R}^n)$ |

| | |
|---|---|
| supp $f$ | support of a function $f$, supp $f = \overline{\{x \mid f(x) \neq 0\}}$ |
| $f_E$ | mean value of an integrable function $f : \mathbb{R}^n \to \mathbb{R}^m$ over $E \subset \mathbb{R}^n$, $f_E = \frac{1}{\lvert E\rvert} \int_E f \, \mathrm{d}x$ |
| $\mathcal{F}f$ | Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$, $(\mathcal{F}f)(\varphi) = f(\mathcal{F}\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where the second $\mathcal{F}$ denotes the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ |
| $\mathcal{F}^{-1}f$ | inverse Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ |
| $f^*$ | Legendre transform of $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ |
| $f * g$ | convolution of $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, $(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x-y) \, \mathrm{d}y$ |
| $\lambda_f$ | distribution function of a measurable function $f : \Omega \to \mathbb{R}^m$, see (5.19) |
| $lip(f;\Sigma)$ | Lipschitz constant of a Lipschitz function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ on $\Sigma \subset \mathbb{R}^{m\times n}$ |
| $\operatorname{osc}(f;\Sigma)$ | oscillation of $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ on $\Sigma \subset \mathbb{R}^{m\times n}$, see Lemma 5.2 |
| $sc^- f$ | lower semicontinuous envelope of $f : X \to \overline{\mathbb{R}}$, $X$ metric space, see Definition 3.7 |
| $f^{\mathrm{c}}$ | convex envelope of $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ |
| $f^{\mathrm{qc}}$ | quasiconvex envelope of $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}}$, see Definition 3.10 |
| $f^{\mathrm{pc}}$ | polyconvex envelope of $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}}$, see Definition 3.10 |
| $f^{\mathrm{rc}}$ | rank-one convex envelope of $f : \mathbb{R}^{m\times n} \to \overline{\mathbb{R}}$, see Definition 3.10 |

*Derivatives*

| | |
|---|---|
| $\partial_j u$, $\frac{\partial u}{\partial x_j}$ | (weak) partial derivative of $u : \Omega \to \mathbb{R}^m$ with respect to the $j$-th component $x_j$ |
| $Du$ | differential of $u : \Omega \to \mathbb{R}^m$, $Du(x) : \mathbb{R}^n \to \mathbb{R}^m$ linear mapping for almost all $x \in \Omega$ |
| $\nabla u$ | gradient of $u : \Omega \to \mathbb{R}^m$, $(\nabla u)_{ij} = \partial_j u_i$ for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ |
| $\operatorname{div} u$ | divergence of $u : \Omega \to \mathbb{R}^n$, $\operatorname{div} u = \sum_{i=1}^n \partial_i u_i$ |
| $\operatorname{curl} u$ | curl of $u : \Omega \to \mathbb{R}^n$, antisymmetric tensor of order two, $(\operatorname{curl} u)_{ij} = \partial_i u_j - \partial_j u_i$ for $i, j \in \{1, \dots, n\}$ |
| $\Delta u$ | Laplacian of $u : \Omega \to \mathbb{R}^m$, $(\Delta u)_i = \sum_{j=1}^n \partial_j \partial_j u_i$ for $i \in \{1, \dots, m\}$ |
| $\operatorname{Det} \nabla u$ | distributional determinant of $\nabla u$ for $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, $\operatorname{Det} \nabla u = \partial_2(u_2\, \partial_1 u_1) - \partial_1(u_2\, \partial_2 u_1)$ |
| $\partial f$ | subdifferential of $f : V \to \overline{\mathbb{R}}$, $V$ normed vector space, $\partial f(v) = \{v' \in V' \mid f(w) \geq f(v) + \langle v', w - v \rangle \ \ \forall w \in V\}$, $v \in V$ |
| $f'$ | classical first derivative of a differentiable function $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$ |
| $\dot{f}$ | partial derivative of the time-continuous function $f : [0, T] \times \Omega \to \mathbb{R}^m$ or $f : [0, T] \times \Omega \to \mathbb{R}^{m \times n}$ with respect to time, $t \in [0, T]$ time variable |
| $\nabla f$ | gradient of the time-continuous function $f : [0, T] \times \Omega \to \mathbb{R}^m$ with respect to the space variables, $t \in [0, T]$ time variable |
| $\frac{\partial}{\partial F} f$ | partial derivative of $f : \mathbb{R}^{n \times n} \times \mathbb{R}^m \to \mathbb{R}$, $(F, x) \mapsto f(F, x)$ with respect to the first entry $F$ |
| $\frac{\partial}{\partial x} f$ | partial derivative of $f : \mathbb{R}^{n \times n} \times \mathbb{R}^m \to \mathbb{R}$, $(F, x) \mapsto f(F, x)$ with respect to the second entry $x$ |

*Convergence*

| | |
|---|---|
| $\to$ | strong convergence |
| $\rightharpoonup$ | weak convergence |
| $\overset{*}{\rightharpoonup}$ | weak-$*$ convergence |
| $\xrightarrow{\square}$ | convergence in the sense of Definition 6.19 |
| $\xrightarrow{\Gamma}$ | $\Gamma$-convergence, defined in Definition 3.19 |

*Modeling elastoplasticity*

| | |
|---|---|
| $n$ | space dimension, $n = 2, 3$ |
| $\Omega$ | reference configuration of an elastoplastic body, $\Omega \subset \mathbb{R}^n$ |
| $u$ | total time-dependent deformation, $u : [0, T] \times \Omega \to \mathbb{R}^n$ with $T > 0$ |
| $F$ | deformation gradient $F = \nabla u$,<br>multiplicative decomposition $F = F_{\mathrm{el}} F_{\mathrm{pl}}$ |
| $F_{\mathrm{el}}$ | elastic part of the deformation gradient |
| $F_{\mathrm{pl}}$ | plastic part of the deformation gradient |
| $P$ | inverse of $F_{\mathrm{pl}}$, $P = F_{\mathrm{pl}}^{-1}$ |
| $z = (P, p)$ | set of internal plastic variables |
| $p$ | hardening parameters |
| $v$ | time-dependent displacement, $v : [0, T] \times \Omega \to \mathbb{R}^n$ with $T > 0$ |
| $\varepsilon$ | strain tensor in linearized elastoplasticity, $\varepsilon = \frac{1}{2}\left(\nabla v + (\nabla v)^T\right)$,<br>additive decomposition $\varepsilon = \varepsilon_{\mathrm{el}} + \varepsilon_{\mathrm{pl}}$ |
| $\varepsilon_{\mathrm{el}}$ | elastic part of the strain tensor |
| $\varepsilon_{\mathrm{pl}}$ | plastic part of the strain tensor |
| $\psi$, $\overline{\psi}$ | internal stored energy density |
| $T$ | first Piola-Kirchhoff stress tensor |
| $Q$ | conjugate plastic stresses |
| $q$ | conjugate hardening forces |
| $\varphi$, $\overline{\varphi}$ | yield function |
| $\mathbb{Q}$ | set of admissible stresses |
| $\mathcal{E}$ | stored energy functional |
| $l$ | time-dependent external loading, $\langle l(t), u \rangle = \int_\Omega f(t)u \,\mathrm{d}x + \int_{\partial\Omega} g(t)u \,\mathrm{d}S$<br>with $f$ and $g$ the applied body and surface forces, $\int_{\partial\Omega} h \,\mathrm{d}S$ the integral<br>of $h$ over $\partial\Omega$ with respect to the $(n-1)$-dimensional surface measure |
| $\mathcal{D}$ | total dissipation |
| $D$, $\overline{D}$, $\hat{D}$ | dissipation distance |
| $\Delta$, $\overline{\Delta}$ | dissipation potential |
| $\psi_{\mathrm{h}}$ | axially symmetric hardening energy density |
| $\psi_{\mathrm{el}}$ | elastic part of the stored energy density |
| $\psi_{\mathrm{cond}}$, $\hat{\psi}_{\mathrm{cond}}$ | condensed or reduced energy density |

*Single-slip models*

| | |
|---|---|
| $(s, m)$ | slip system, $s$ slip direction, $m$ slip plane normal, $s \perp m$, $\lvert s\rvert = \lvert m\rvert = 1$ |
| $\gamma$ | amount of slip along $(s, m)$ |
| $(e_1, e_2)$ | slip system in the 3D models, simplification $s = e_1$, $m = e_2$ |
| $W_{\mathrm{el}}$ | elastic energy density depending on $F_{\mathrm{el}}$ |
| $W_{\mathrm{pl}}$ | plastic energy density depending on $F_{\mathrm{pl}}$ |
| Diss | dissipated energy density due to plastic deformation depending on $F_{\mathrm{pl}}$ |
| $W_{\mathrm{cond}}$ | condensed energy density |
| $W_p$ | condensed energy density of the elastically rigid model, $p \geq 1$ plastic exponent, see (4.8) |
| $W$ | $W_p$ in case the applied plastic exponent is clear from the context |
| $W_{\mathrm{el},\varepsilon;\,q}$ | elastic energy density of the considered single-slip model, $\varepsilon > 0$, $q \geq 1$ elastic exponent, see (4.16) |
| $W_{\mathrm{el},\varepsilon}$ | $W_{\mathrm{el},\varepsilon;\,q}$ in case the applied elastic exponent is clear from the context |
| $W_{\varepsilon;\,q,p}$ | condensed energy density of the single-slip model with elastic energy, $q \geq 1$ elastic exponent, $p \geq 1$ plastic exponent, see (4.17) |
| $W_\varepsilon$ | $W_{\varepsilon;\,q,p}$ in case the applied elastic and plastic exponents are clear from the context |
| $\mathcal{M}^{(2)}$ | subset of $\mathbb{R}^{2\times 2}$ where $W$ in the two-dimensional setting is finite, see (4.3) and (4.4) |
| $\mathcal{M}^{(3)}$ | subset of $\mathbb{R}^{3\times 3}$ where $W$ in the three-dimensional setting is finite, see (4.3) and (4.5) |
| $\mathcal{N}^{(2)}$ | subset of $\mathbb{R}^{2\times 2}$ where $W^{\mathrm{qc}}$ in the two-dimensional setting is finite, $\mathcal{N}^{(2)} = (\mathcal{M}^{(2)})^{\mathrm{rc}}$ |
| $\mathcal{N}^{(3)}$ | subset of $\mathbb{R}^{3\times 3}$ where $W^{\mathrm{rc}}$ in the three-dimensional setting is finite, $\mathcal{N}^{(3)} = (\mathcal{M}^{(3)})^{\mathrm{rc}}$ |
| $E_{\varepsilon;q,p}$ | energy functional of the model with elastic energy, $\varepsilon > 0$, see (7.7) and Theorem 8.3 |
| $E_\varepsilon$ | $E_{\varepsilon;q,p}$ in case the applied parameters are clear from the context |
| $E_p$ | energy functional of the rigid model defined in (7.8) and Theorem 8.3 |
| $E$ | $E_p$ in case the applied plastic exponent is clear from the context |

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